Statistics and Probability Letters 89 (2014) 131–137

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Fractional telegraph-type equations and hyperbolic Brownian motion

Mirko D'Ovidio [a,*], Enzo Orsingher [b], Bruno Toaldo [b]

[a] Department of Basic and Applied Sciences for Engineering, Sapienza University of Rome, Italy
[b] Department of Statistical Sciences, Sapienza University of Rome, Italy

ARTICLE INFO

Article history:
Received 20 December 2013
Received in revised form 21 February 2014
Accepted 24 February 2014
Available online 12 March 2014

MSC:
60G51
60G52
35C05

Keywords:
Riemann–Liouville fractional calculus
Hyperbolic Brownian motion
Telegraph processes
Subordinators
Time-changed processes
Hyperbolic Laplacian

ABSTRACT

This paper is devoted to the interplay between time-fractional telegraph-type equations and processes defined on the n-dimensional Poincaré half-space $\mathbb{H}^n$. We solve such equations and show that the solutions coincide with the law of the composition of a hyperbolic Brownian motion with the inverse of the sum of two independent stable subordinators. In the case $n = 3$, we obtain the explicit form of the solution of the above equation.

1. Introduction

This paper is concerned with random motions on the hyperbolic Poincaré half-space $\mathbb{H}^n = \left\{\boldsymbol{x}, y : \boldsymbol{x} \in \mathbb{R}^{n-1}, y > 0\right\}$. We focus our attention on the behaviour of the hyperbolic random distance η of $(\boldsymbol{x}, y)$ from the origin $O = (\mathbf{0}, 1)$ which is given by

$$\cosh \eta = \frac{\sum_{j=1}^{n-1} x_i^2 + y^2 + 1}{2y}. \tag{1.1}$$

We consider the random motions in $\mathbb{H}^n$ where the distance η has distribution $p_n^\nu(\eta, t)$ satisfying the telegraph-type fractional equation of the form

$$\begin{cases} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu}\right) p_n^\nu(\eta, t) = \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\left(\frac{1}{\sinh^{n-1}\eta} p_n^\nu(\eta, t)\right)\right), & \eta > 0, t > 0 \\ p_n^\nu(\eta, 0) = \delta(\eta), \end{cases} \tag{1.2}$$

* Corresponding author. Tel.: +39 0649766705.
E-mail addresses: mirko.dovidio@uniroma1.it (M. D'Ovidio), enzo.orsingher@uniroma1.it (E. Orsingher), bruno.toaldo@uniroma1.it (B. Toaldo).

http://dx.doi.org/10.1016/j.spl.2014.02.021
0167-7152/

for $0 < \nu \le \frac{1}{2}$ and $n \in \mathbb{N}$. The kernel $\kappa_n^\nu(\eta, t)$ corresponding to $p_n^\nu(\eta, t)$ defined as

$$\kappa_n^\nu(\eta, t) = \frac{1}{\sinh^{n-1}\eta} p_n^\nu(\eta, t), \qquad \eta > 0, t > 0, \tag{1.3}$$

solves instead the fractional equation

$$\begin{cases} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu}\right) \kappa_n^\nu(\eta, t) = \frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta} \left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta} \kappa_n^\nu(\eta, t)\right), & \eta > 0, t > 0, \\ \kappa_n^\nu(\eta, 0) = \delta(\eta). \end{cases} \tag{1.4}$$

where the operator

$$\mathcal{G}^\star = \frac{\partial}{\partial \eta} \left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta} \left(\frac{1}{\sinh^{n-1}\eta}\right)\right), \qquad n \in \mathbb{N} \tag{1.5}$$

is the adjoint of

$$\mathcal{G} = \frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta} \left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\right). \tag{1.6}$$

The fractional derivatives in (1.2) and (1.4) must be understood in the sense of Dzerbayshan–Caputo, that is for $\nu \in (0, 1)$

$$\frac{\partial^\nu}{\partial t^\nu} u(x, t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{\frac{\partial}{\partial s} u(x, s)}{(t-s)^\nu} ds, \quad \nu \in (0, 1). \tag{1.7}$$

For Dzerbayshan–Caputo fractional calculus consult, for example, Kilbas et al. (2006). The process $\mathcal{T}_n^\nu(t)$, $t > 0$, in $\mathbb{H}^n$ which possesses distribution $p_n^\nu(x, t)$ solving (1.2) is obtained by means of the composition

$$\mathcal{T}_n^\nu(t) = B_n^{\mathrm{hp}}\left(\mathcal{L}^\nu(t)\right), \quad t > 0, \tag{1.8}$$

where B_n^{hp} is the hyperbolic Brownian motion in $\mathbb{H}^n$ independent from $\mathcal{L}^\nu(t)$ which is defined as

$$\mathcal{L}^\nu(t) = \inf\left\{s > 0 : H_1^{2\nu}(s) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(s) \ge t\right\}, \quad \nu \in \left(0, \frac{1}{2}\right], \tag{1.9}$$

for $H_1^{2\nu}, H_2^\nu$ independent stable subordinators. The hyperbolic Brownian motion has been introduced in the plane by Gertsenshtein and Vasiliev (1959) and in $\mathbb{H}^3$ by Karpelevich et al. (1959). In successive papers many properties of the hyperbolic Brownian motion have been explored (see for example Getoor (1961), Gruet (1996), Lao and Orsingher (2007) and Matsumoto and Yor (2005)). The relationship between kernels in $\mathbb{H}^2$ and $\mathbb{H}^3$ and kernels in higher-order spaces is represented by the Millson formula

$$k_{n+2}(\eta, t) = -\frac{e^{-nt}}{2\pi \sinh\eta} \frac{\partial}{\partial \eta} k_n(\eta, t), \quad \eta > 0, t > 0, n \in \mathbb{N}. \tag{1.10}$$

Since p_3^{hp} and k_3 are considerably simpler than p_2^{hp} and k_2 we give explicit expressions for the distribution

$$p_3^{\frac{1}{2}}(\eta, t) = \frac{\lambda \eta \sinh\eta}{2\pi} \int_0^t \frac{e^{-s}}{s^{\frac{3}{2}}\sqrt{t-s}} e^{-\frac{\lambda^2 s^2}{t-s} - \frac{\eta^2}{4s}} \left(\frac{s}{t-s} + 2\right) ds, \tag{1.11}$$

where $\eta > 0$ and $t > 0$ and $p_3^{\frac{1}{2}}$ is the probability density of $B_3^{\mathrm{hp}}\left(\mathcal{L}^{\frac{1}{2}}(t)\right)$, $t > 0$. This distribution solves the fractional-hyperbolic telegraph equation (1.2), for $\nu = \frac{1}{2}$ and $n = 3$.

2. Hyperbolic fractional telegraph equations

The hyperbolic Brownian motion is a diffusion on the Poincaré half-space

$$\mathbb{H}^n = \left\{(\boldsymbol{x}, y) : \boldsymbol{x} \in \mathbb{R}^{n-1}, y > 0\right\}, \tag{2.1}$$

with generator, written in cartesian coordinates,

$$\mathfrak{H}_n = \frac{1}{2}\left[y^2 \left(\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y^2}\right) + (2-n) y \frac{\partial}{\partial y}\right]. \tag{2.2}$$