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Estimates for the overshoot of a random walk with negative drift and non-convolution equivalent increments[☆]

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ABSTRACT

Let $\{S_n : n \geq 0\}$ be a random walk with negative drift and $\tau(x)$ be the first time when the random walk crosses a given level $x \geq 0$. This paper focuses on random walks with non-convolution equivalent increments. For this random walk, the uniform asymptotics of $P(S_{\tau(x)} - x > y, \tau(x) < \infty)$, as $x \to \infty$, have been presented.

1. Introduction

Let $\{X_n : n \geq 1\}$ be independent identically distributed (i.i.d.) random variables (r.v.s) with common distribution F on $(-\infty, \infty)$ satisfying $F(x) < 1$ for all $x \in (-\infty, \infty)$. Let $\{S_n : n \geq 0\}$ be a random walk generated by $\{X_n : n \geq 1\}$, that is, $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$ with $S_0 \equiv 0$, and let $M = \sup\{S_n : n \geq 0\}$ be the supremum of the random walk. Let W be the distribution of M. Denote by

$$\tau(x) = \inf\{n \geq 1 : S_n > x\}, \quad x \geq 0,$$

the first time when the random walk $\{S_n : n \geq 0\}$ crosses a given level x, with the convention $\inf \phi = \infty$. Let

$$A(x) = S_{\tau(x)} - x$$

be the overshoot of the random walk at level x.

The overshoot is one of the fundamental objects of random walks and it has been investigated by many researchers. We refer the reader to Janson (1986), Chang (1994), Asmussen and Klüppelberg (1996), Borovkov and Foss (2000), Cheng et al.

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(2002), Tang (2007), Cui et al. (2009), Chen et al. (2009), Tang and Wei (2010), Wang and Wang (2011), Wang et al. (2011) and Wang (2013), among others. In this paper, we are interested in the uniform asymptotics of the tail probability of the overshoot $A(x)$ for non-convolution equivalent increments. In order to better illustrate our motivation and results, we will introduce some notions and notation.

Without special statement, in this paper a limit is taken as $x \to \infty$. For positive functions $a_i(\cdot)$ and $b_i(\cdot,\cdot)$, $i = 1, 2$, we write $a_1(x) \lesssim a_2(x)$ if $\limsup_{x\to\infty} a_1(x)/a_2(x) \le 1$, write $a_1(x) \gtrsim a_2(x)$ if $\liminf_{x\to\infty} a_1(x)/a_2(x) \ge 1$, write $a_1(x) \sim a_2(x)$ if $\lim_{x\to\infty} a_1(x)/a_2(x) = 1$, and write $a_1(x) = o(1)a_2(x)$ if $\lim_{x\to\infty} a_1(x)/a_2(x) = 0$. Moreover, we say that $b_1(x, y) \lesssim b_2(x, y)$, as $x \to \infty$, holds uniformly for y in some nonempty set Δ if

$$\limsup_{x\to\infty} \sup_{y\in\Delta} \frac{b_1(x, y)}{b_2(x, y)} \le 1;$$

say that $b_1(x, y) \gtrsim b_2(x, y)$, as $x \to \infty$, holds uniformly for y in some nonempty set Δ if

$$\liminf_{x\to\infty} \inf_{y\in\Delta} \frac{b_1(x, y)}{b_2(x, y)} \ge 1;$$

and say that $b_1(x, y) \sim b_2(x, y)$, as $x \to \infty$, holds uniformly for y in some nonempty set Δ if $b_1(x, y) \lesssim b_2(x, y)$ and $b_1(x, y) \gtrsim b_2(x, y)$ both hold uniformly for y in the set Δ.

For a distribution V on $(-\infty, \infty)$, the tail of V is $\overline{V} = V(\infty) - V$, and let $V_I(x) = \left(\int_0^\infty \overline{V}(u)du\right)^{-1} \int_0^x \overline{V}(u)du$, $x \ge 0$, be the integrated tail distribution of V. Let V^{n*} be the n-fold convolution of V for $n = 0, 1, 2, \ldots$, where $V^{1*} = V$ and V^{0*} is the distribution degenerated at zero. For a real number γ, denote by $\widehat{V}(\gamma) = \int_{-\infty}^{\infty} e^{\gamma u}V(du)$ the moment-generating function of V.

In the following, we will introduce some distribution classes. Let V be a distribution on $(-\infty, \infty)$. We say that the distribution V belongs to the class $\mathscr{L}(\gamma)$, for some $\gamma \ge 0$, if, for any $t \in (-\infty, \infty)$,

$$\overline{V}(x - t) \sim e^{\gamma t}\overline{V}(x),$$

where, when $\gamma > 0$ and V is a lattice distribution, x and t are both taken as a multiple of the lattice step. We say that the distribution V belongs to the class $\mathscr{S}(\gamma)$ for some $\gamma \ge 0$ if $V \in \mathscr{L}(\gamma)$, $\widehat{V}(\gamma) < \infty$ and

$$\overline{V^{2*}}(x) \sim 2\widehat{V}(\gamma)\overline{V}(x).$$

The class $\mathscr{S}(\gamma)$, $\gamma \ge 0$ is called the convolution equivalent distribution class; it was introduced by Chistyakov (1964) and Chover et al. (1973a,b) for distributions on $[0, \infty)$, and by Pakes (2004) for distributions on $(-\infty, \infty)$. In particular, we call $\mathscr{S}(0)$ and $\mathscr{L}(0)$ the subexponential distribution class and the long-tailed distribution class, denoted by $\mathscr{S}$ and $\mathscr{L}$, respectively.

This paper will mainly investigate the case that the distributions of the increments of random walks may not be convolution equivalent. So we will need another distribution class. We say that the distribution V belongs to the class $\mathscr{OS}$ if $\overline{V}(x) > 0$ for sufficiently large x and

$$C_V^* = \limsup_{x\to\infty} \frac{\overline{V^{2*}}(x)}{\overline{V}(x)} < \infty.$$

Clearly, if $V \in \mathscr{S}(\gamma)$ for some $\gamma \ge 0$, then $C_V^* = 2\widehat{V}(\gamma)$. Therefore, for each $\gamma \ge 0$, $\mathscr{S}(\gamma) \subset \mathscr{OS}$. If $V \in \mathscr{L}(\gamma)$ for some $\gamma \ge 0$, then $C_V^* \ge 2\widehat{V}(\gamma)$, which can be obtained by Lemma 2.4 of Embrechts and Goldie (1982) and Theorems 1.1 and 1.2 of Yu et al. (2010). The class $\mathscr{OS}$ was first introduced by Klüppelberg (1990) and studied in detail by Klüppelberg and Villasenor (1991), Shimura and Watanabe (2005), Watanabe and Yamamura (2010), Lin and Wang (2012) and Yang and Wang (2011), among others.

In the following, we will introduce some other notions and notation about random walks. Recalling the definition of $\tau(x)$, $\tau_+ = \tau(0)$ is the first ascending ladder-epoch, and let F_+ be the distribution of S_{τ_+}. Similarly, the first (weak) descending ladder-epoch is defined as

$$\tau_- = \inf\{n \ge 1 : S_n \le 0\},$$

and let F_- be the distribution of S_{τ_-}. Throughout this paper, we assume that F has a finite mean $-\mu < 0$. Hence (see, e.g., Asmussen (2003, Chap. VIII)), S_n drifts to $-\infty$, M is finite almost surely, $\tau(x)$ is defective, i.e., $P(\tau(x) < \infty) < 1$ for all $x \ge 0$, and F_- is proper, but F_+ is defective and

$$q = F_+(\infty) = 1 - e^{-B} < 1;$$

here, $B = \sum_{n=1}^{\infty} n^{-1}P(S_n > 0) < \infty$. Let $G(x) = P(S_{\tau_+} \le x|\tau_+ < \infty) = q^{-1}F_+(x)$, $x \ge 0$; then G is proper.

When the distributions (or related distributions) of the increments of random walks belong to the convolution equivalent distribution class, Tang (2007) and Wang (2013) have obtained the uniform asymptotics of the tail probability of the overshoot $A(x)$.