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On comparison of reversed hazard rates of two parallel systems comprising of independent gamma components

Neeraj Misra [a], Amit Kumar Misra [b,*]

[a] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur-208016, India

[b] Department of Statistics, Central University of Bihar, Patna-800014, India

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ABSTRACT

Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be independent random variables such that X_i (Y_i) follows the gamma distribution with shape parameter α and mean $\frac{\alpha}{\lambda_i}$ $(\frac{\alpha}{\mu_i})$, $\alpha > 0$, $\lambda_i > 0$ $(\mu_i > 0)$, $i = 1, \ldots, n$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and let $\tilde{r}_{n:n}(\boldsymbol{\lambda}; x)$ $(\tilde{r}_{n:n}(\boldsymbol{\mu}; x))$ denote the reversed hazard rate of $\max\{X_1, \ldots, X_n\}$ $(\max\{Y_1, \ldots, Y_n\})$. In this note we show that if $\boldsymbol{\lambda}$ weakly majorizes $\boldsymbol{\mu}$ then $\tilde{r}_{n:n}(\boldsymbol{\lambda}; x) \geq \tilde{r}_{n:n}(\boldsymbol{\mu}; x)$, $\forall x > 0$, thereby strengthening the results of Dykstra et al. (1997), and Lihong and Xinsheng (2005).

1. Introduction

Let $X_1, \ldots, X_n$ be independent and nonnegative random variables (i.e. corresponding distributions have the common support $\mathbb{R}_+ \equiv [0, \infty)$) representing the lifetimes of n components and let $Y_1, \ldots, Y_n$ be another set of independent and nonnegative random variables representing the lifetimes of another set of n components. For $k \in \{1, \ldots, n\}$, let $X_{k:n}$ and $Y_{k:n}$ respectively denote the kth order statistics based on random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Then $X_{k:n}$ and $Y_{k:n}$ are the lifetimes of $(n - k + 1)$-out-of-n systems constructed from the two sets of components and thus a stochastic comparison of these two random variables may be of interest. A vast literature on stochastic comparisons of order statistics from two heterogeneous distributions is available. See, for example, Pledger and Proschan (1971), Proschan and Sethuraman (1976), Boland et al. (1994), Hu (1995), Dykstra et al. (1997), Khaledi and Kochar (2007), Kochar and Xu (2007a,b), Zhao and Balakrishnan (2011), Khaledi et al. (2011), and references therein. In order to provide a brief review of the literature on this topic we will require definitions of some stochastic orders, aging classes, and the concept of majorization. For definitions and properties of various stochastic orders and aging classes, one may refer to Shaked and Shanthikumar (2007), and Barlow and Proschan (1975). Readers may refer to Marshall and Olkin (1979) and Bon and Păltănea (1999) for comprehensive details of majorization and p-larger order.

Suppose that the random variables X_i and Y_i have absolutely continuous distribution functions $F(x; \lambda_i)$ and $F(x; \mu_i)$, respectively, where $\lambda_i, \mu_i > 0$, $i = 1, \ldots, n$. Let $\bar{F}(x; \lambda_i) = 1 - F(x; \lambda_i)$ and $\bar{F}(x; \mu_i) = 1 - F(x; \mu_i)$ be the corresponding survival functions. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$.

* Corresponding author. Tel.: +91 9839425105; +91 7488466681(Mob.).
E-mail addresses: neeraj@iitk.ac.in (N. Misra), amishra@cub.ac.in, mishraamit31@gmail.com (A.K. Misra).

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Under the proportional hazard rates (PHR) model (i.e. $\bar{F}(x;\lambda) = [\bar{F}_0(x)]^{\lambda}$, $x \in \mathbb{R} \equiv (-\infty, \infty)$, $\lambda > 0$, for some survival function $\bar{F}_0$), Pledger and Proschan (1971) proved that

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{k:n} \le_{\mathrm{st}} X_{k:n}, \quad k = 1, \ldots, n. \tag{1.1}$$

Proschan and Sethuraman (1976) strengthened this result from componentwise stochastic ordering to multivariate stochastic ordering. For two-component parallel systems, Boland et al. (1994) strengthened result (1.1) by showing that $(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\mu_1, \mu_2)$ implies $Y_{2:2} \le_{\mathrm{hr}} X_{2:2}$. Using an example they also demonstrated that this result may not hold for $n \ge 3$ component parallel systems. However, for $n \ge 2$ component parallel systems with exponentially distributed lifetimes (i.e. $\bar{F}_0(x) = e^{-x}$, $x \in \mathbb{R}_+$), Dykstra et al. (1997) showed that

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\mathrm{rh}} X_{n:n}. \tag{1.2}$$

For parallel systems, Khaledi and Kochar (2006) generalized result (1.1) in another direction by establishing that $\boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\mu}$ implies $Y_{n:n} \le_{\mathrm{st}} X_{n:n}$. Using an example they demonstrated that this result may not hold for other order statistics.

For systems with gamma distributed lifetimes (i.e. $F(x;\lambda) = \int_0^{\lambda x} t^{\alpha-1} e^{-t}\, \mathrm{d}t / \Gamma(\alpha)$, $x, \lambda, \alpha > 0$), Lihong and Xinsheng (2005) proved that

$$\alpha > 1 \quad \text{and} \quad \boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow X_{1:n} \le_{\mathrm{st}} Y_{1:n}; \tag{1.3}$$

$$\alpha \le 1 \quad \text{and} \quad \boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow (Y_{1:n}, \ldots, Y_{n:n}) \le_{\mathrm{st}} (X_{1:n}, \ldots, X_{n:n}); \tag{1.4}$$

$$\forall \alpha > 0, \quad \boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\mathrm{st}} X_{n:n}. \tag{1.5}$$

In this paper we continue the study on stochastic comparisons of order statistics from heterogeneous gamma distributions further by generalizing result (1.2) from the exponential case to the gamma case. Specifically, in Section 2 of the paper, we show that

$$\forall \alpha > 0, \quad \boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\mathrm{rh}} X_{n:n}.$$

This result may also be viewed as an extension of results (1.3)–(1.4), and generalization of result (1.5).

Throughout the paper, when we say that a function is increasing (decreasing) it means that the function is non-decreasing (non-increasing). Moreover all the distributions under study shall be assumed to be absolutely continuous with support $\mathbb{R}_+$. For any probability density function h, we will assume that $\{x \in \mathbb{R} : 0 < h(x) < 1\} = \mathbb{R}_+$.

2. Comparison of reversed hazard rates

Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be independent gamma random variables with X_i (Y_i) having probability density function $f(x;\alpha,\lambda_i)$ $(f(x;\alpha,\mu_i))$, $\lambda_i > 0$ $(\mu_i > 0)$, $i = 1, \ldots, n$, where for $\lambda > 0$ and $\alpha > 0$,

$$f(x;\alpha,\lambda) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0, \\ 0 & \text{otherwise}. \end{cases}$$

To prove the main result we need the following lemmas. The proof of Lemma 2.1 is straightforward and hence omitted.

Lemma 2.1. *Let W be a random variable having the probability density function*

$$h(u;\alpha,y) = \begin{cases} \dfrac{(1-u)^{\alpha-1}\, e^{yu}}{\int_0^1 (1-t)^{\alpha-1}\, e^{yt}\, \mathrm{d}t} & \text{if } 0 < u < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

where α and y are given positive constants. Then, W has the increasing hazard rate (IHR).

Lemma 2.2 (*Barlow and Proschan, 1975, p. 118*)**.** *Let X be a nonnegative random variable with distribution function F and let $\mu_r = \int_0^{\infty} x^r\, \mathrm{d}F(x)$, $r = 1, 2$. If X has the increasing hazard rate in average (IHRA) then $\mu_2 \le 2\mu_1^2$.*

Theorem 2.1. *For any $\alpha > 0$ and $n \ge 2$,*

$$\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \le_{\mathrm{rh}} X_{n:n}.$$