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Large deviations for a class of counting processes and some statistical applications

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ABSTRACT

The aim of this paper is to prove results on large deviations for a class of counting processes, and to illustrate some statistical applications. We also present a generalization of the results for a class of compound processes. The statistical applications describe the asymptotic behavior of some issues concerning two hypothesis testing problems, and the logarithmic rates are expressed in terms of the large deviation rate functions.

1. Introduction

The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale; see e.g. Dembo and Zeitouni (1998) as a reference on this topic. In this paper we prove results on large deviations for families of counting processes (with some statistical applications) and of compound processes.

The main results concern a class of counting processes $\{N_\beta(\cdot) : \beta > 0\}$; see Eq. (2), where $\{N_\beta : \beta > 0\}$ and $\{X_n : n \geq 1\}$ are the random variables in the conditions $(\mathbf{N_1})$, $(\mathbf{X_1})$ and $(\mathbf{I_1})$. Then we can say that, for each fixed $\beta > 0$, we have $N_\beta = N_\beta(1)$ and, conditionally on the event $\{N_\beta = n\}$ (for some $n \geq 1$), the occurrence times of $N_\beta(\cdot)$ are the increasing order statistics of $X_1, \ldots, X_n$. Thus we have the so-called *order statistics property* (see e.g. Example 4.1.7 in Embrechts et al. (1997) where this property is stated for the Poisson process); a characterization and a generalization of this property can be found in Crump (1975) and Debrabant (2012), respectively. We recall the paper Balakrishnan and Kozubowski (2008), where the random variables $\{X_n : n \geq 1\}$ are i.i.d. uniformly distributed on $[0, 1]$ and N_β is a weighted Poisson distributed random variable (thus $N_\beta(\cdot)$ is a weighted Poisson process); the interested reader can see Borges et al. (2012) for a generalization where the random variables $\{X_n : n \geq 1\}$ are pairwise equicorrelated. Here we consider a class of examples where N_β is a weighted Poisson distributed random variable and, in particular, the weights of the alternative version of the fractional Poisson process in Beghin and Macci (2013).

An interesting contribution of this paper is given by the consequences of the main results for two hypothesis testing problems; namely we mean the asymptotic behavior of the likelihood ratio test statistics (for the first one) and of the

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first kind error probabilities (for the second one). In both cases the logarithmic asymptotic rates are expressed in terms of sample path large deviation rate functions. Moreover, according to the terminology in the literature on large deviations (see e.g. Lemma 4.2 in Ganesh et al., 2004; see also Mandjes, 2007, p. 45), we determine a most likely path leading to the first kind error (see Remark 3.5).

Some further minor results concern a class of compound processes $\{A_\beta(\cdot) : \beta > 0\}$; see (10), where $\{N_\beta : \beta > 0\}$ and $\{(B_n, X_n) : n \geq 1\}$ are the random variables in the conditions $(\mathbf{N_2})$, $(\mathbf{X_2})$ and $(\mathbf{I_2})$. Obviously, when N_β is weighted Poisson distributed, $A_\beta(1)$ is a compound weighted Poisson distributed, and we recall the recent paper Minkova and Balakrishnan (2013) where compound weighted Poisson distributions are studied; moreover, if we refer to the insurance terminology, when the random summands $\{B_n : n \geq 1\}$ are positive and independent of $\{X_n : n \geq 1\}$, $A_\beta(\cdot)$ coincides with the aggregate claim process in Lefèvre and Picard (2011, Section 3). We remark that the results for the compound processes generalize the ones cited above for the counting processes, but we cannot provide an explicit formula for the sample path large deviation rate functions as we do for $\{N_\beta(\cdot) : \beta > 0\}$ under suitable conditions.

We conclude with the outline of the paper. We start with some preliminaries in Section 2. In Section 3 we present the results and the statistical applications for the family of counting processes. Section 4 is devoted to the compound processes. We also present a final Appendix for the proof of formula (7).

2. Preliminaries

We start with some basic definitions on large deviations (see e.g. Dembo and Zeitouni, 1998, pp. 4–5). Let $\mathcal{Z}$ be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}_{\mathcal{Z}}$. A lower semi-continuous function $I : \mathcal{Z} \to [0, \infty]$ is called rate function. A family of $\mathcal{Z}$-valued random variables $\{Z_\beta : \beta > 0\}$ satisfies the *large deviation principle* (LDP for short), as $\beta \to \infty$, with rate function I if

$$\limsup_{\beta\to\infty} \frac{1}{\beta} \log P(Z_\beta \in C) \leq -\inf_{z\in C} I(z) \quad \text{for all closed sets } C$$

and

$$\liminf_{\beta\to\infty} \frac{1}{\beta} \log P(Z_\beta \in O) \geq -\inf_{z\in O} I(z) \quad \text{for all open sets } O.$$

A rate function I is said to be good if all the level sets $\{\{z \in \mathcal{Z} : I(z) \leq \gamma\} : \gamma \geq 0\}$ are compact.

We often apply the Gärtner–Ellis Theorem (see e.g. Theorem 2.3.6 in Dembo and Zeitouni, 1998; see also Definition 2.3.5 in Dembo and Zeitouni, 1998 for the concept of essentially smooth function) and the following condition plays a crucial role:

- $(\mathbf{N})$: $\{N_\beta : \beta > 0\}$ are nonnegative and integer valued random variables such that we can define the function $\Lambda_N : \mathbb{R} \to (-\infty, \infty]$ via the limit

$$\Lambda_N(\alpha) := \lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}\left[e^{\alpha N_\beta}\right] \quad (\text{for all } \alpha \in \mathbb{R}).$$

Note that, since the random variables $\{N_\beta : \beta > 0\}$ are nonnegative, the function Λ_N is nondecreasing; hence, since $\Lambda_N(0) = 0$, we have $(-\infty, 0] \subset \mathcal{D}_{\Lambda_N}$ where

$$\mathcal{D}_{\Lambda_N} := \{\alpha \in \mathbb{R} : \Lambda_N(\alpha) < \infty\}. \tag{1}$$

The Legendre transform of the function $\Lambda_N : \mathbb{R} \to \mathbb{R}$ in condition $(\mathbf{N})$ is defined by

$$\Lambda_N^*(x) := \sup_{\alpha\in\mathbb{R}}\{\alpha x - \Lambda_N(\alpha)\} \quad (\text{for } x \in \mathbb{R}).$$

Throughout the paper we use the notation $\underline{a}_h = (a_1, \ldots, a_h)$ for the vectors in $\mathbb{R}^h$; in some cases we use the same notation (with capital letters) for random vectors. We use the symbol $\langle\cdot,\cdot\rangle$ for the inner product in some $\mathbb{R}^h$. Moreover we always set $0 \log 0 = 0$.

3. On the family of counting processes

In this section we deal with a class of counting processes $\{N_\beta(\cdot) : \beta > 0\}$ on $[0, 1]$ defined below. We consider the following conditions, and we refer to the random variables $\{N_\beta : \beta > 0\}$ in condition $(\mathbf{N})$:

- $(\mathbf{X_1})$: $\{X_n : n \geq 1\}$ are $(0, 1]$-valued i.i.d. random variables;
- $(\mathbf{I_1})$: $\{X_n : n \geq 1\}$ and $\{N_\beta : \beta > 0\}$ are independent.

Then, for each fixed $\beta > 0$, $N_\beta(\cdot) = \{N_\beta(t) : t \in [0, 1]\}$ is defined by

$$N_\beta(t) := \sum_{h=1}^{N_\beta} 1_{\{X_h \leq t\}}. \tag{2}$$

We start with Proposition 3.1, which provides two LDPs: the first one concerns the finite dimensional distributions, the second one concerns the sample paths. An explicit expression of the sample path large deviation rate function $I_{[0,1]}^N$ is