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Intuitive approximations in discrete renewal theory, Part 1: Regularly varying case

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ABSTRACT

It is usually impossible to find explicit expressions for the renewal sequence. This paper presents a simple method to approximate the renewal sequence, which covers many of the known approximations. The paper uses the ideas of Mitov and Omey (2014).

1. Introduction

Suppose that $X, X_1, X_2, \ldots$ are i.i.d. nonnegative integer-valued random variables with p.d.f. $p_k = P(X = k)$, $k \in \mathbb{N}_0$. The d.f. of X is given by $F(x) = P(X \leq x)$ and its tail is denoted by $\overline{F}(x) = 1 - F(x)$. Throughout the paper we assume that (p_k) is aperiodic: $\gcd\{k : p_k > 0\} = 1$. We also assume that $0 < \mu = E(X) < \infty$. For $n \in \mathbb{N}_0$, the partial sums S_n are given by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Note that $P(S_n \leq x) = F^{\star n}(x)$, which is the n-fold convolution of F, i.e. $F^{\star 0} = 1_{[0,\infty)}$ and $F^{\star n} = F \star F^{\star(n-1)}$, where the convolution of two d.f. is defined by $F \star G(x) = \int_0^x F(x-y)dG(y)$. For briefness, we write $F^{\star n}(x)$ as $F_k^{\star n}$ in case x is a nonnegative integer k. Moreover, $P(S_n = k) = p_k^{*n}$, which is the n-fold convolution of (p_k), i.e. $p_k^{*0} = 1_{\{0\}}(k)$ and $p_k^{*n} = (p * p^{*(n-1)})_k$, where the convolution of two sequences (a_k) and (b_k) is defined by $(a * b)_k = \sum_{i=0}^k a_i b_{k-i}$. The generating function of X is $\hat{P}(z) = E(z^X)$, $|z| < 1$, and $\hat{P}(1) = 1$. The generating function of S_n is given by $\hat{P}^n(z)$. Since X has finite expectation, we have $\mu = \hat{P}'(1)$.

Let X_e be a random variable, independent of X, that has the equilibrium distribution corresponding to X, i.e. $p_{e,k} = P(X_e = k) = \overline{F}_k/\mu$ for $k \in \mathbb{N}_0$. The generating function of X_e satisfies $\hat{P}_e(z) = (1 - \hat{P}(z))/(\mu(1-z))$. Define $S_{e,n} = X_{e,1} + \cdots + X_{e,n}$, $p_{e,k}^{*n}$, and $F_{e,k}^{\star n}$ analogously as above.

The renewal sequence (u_n) is defined by $u_n = \sum_{k=0}^{\infty} p_n^{*k}$. The aim of the present paper is to obtain approximations for u_n when n is large. Therefore, all limits that appear later are taken with respect to $n \to \infty$. It is well known that $u_n \to 1/\mu$ and the main problem is to obtain precise estimates for the rate at which $u_n - 1/\mu \to 0$ or $\Delta u_n = u_{n-1} - u_n \to 0$.

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Following the approach of Mitov and Omey (2014), we start from the generating function of (u_n), which is given by $\hat{U}(z) = \sum_{n=0}^{\infty} u_n z^n = (1-\hat{P}(z))^{-1} = (\mu(1-z)(1-(1-\hat{P}_e(z))))^{-1}$. Using a Taylor expansion, we obtain that

$$\hat{U}(z) = \sum_{k=0}^{\infty} \hat{T}_k(z), \quad \text{with} \quad \hat{T}_k(z) = \frac{1}{\mu(1-z)}(1-\hat{P}_e(z))^k. \tag{1}$$

Formula (1) suggests the following approximations $\hat{U}_m(z)$ for $\hat{U}(z)$: $\hat{U}_m(z) = \sum_{k=0}^{m} \hat{T}_k(z)$. By inversion, this approach then leads to approximations $u_{m,n}$ for the renewal sequence u_n of the form $u_{m,n} = \sum_{k=0}^{m} t_{k,n}$, where the sequence $(t_{k,n})$ has generating function $\hat{T}_k(z) = \sum_{n=0}^{\infty} t_{k,n} z^n$. In the next section we will identify $\hat{T}_k(z)$ and $(t_{k,n})$. In this paper we focus on the cases $0 \le m \le 3$ and show that our approximations $(u_{m,n})$ correspond to the approximations that have been published in many papers before.

2. The sequences $(t_{k,n})$ and $(\delta_{k,n})$

2.1. Expressions for $(t_{k,n})$ and $(\delta_{k,n})$

We first identify $\hat{T}_k(z)$. If $k = 0$, then (1) gives $\hat{T}_0(z) = 1/(\mu(1-z))$, which shows that $\hat{T}_0(z)$ is the generating function of $t_{0,n} = 1/\mu$. For $k \ge 1$, the binomial expansion in (1) yields

$$\begin{aligned}\hat{T}_k(z) &= \frac{1}{\mu(1-z)} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \hat{P}_e^i(z) \\ &= \frac{-1}{\mu(1-z)} \sum_{i=1}^{k} \binom{k}{i} (-1)^i (1-\hat{P}_e^i(z)).\end{aligned}$$

Since $\hat{P}_e^i(z)$ is the generating function of $S_{e,i}$, we have $(1-\hat{P}_e^i(z))/(1-z) = \sum_{n=0}^{\infty} \overline{F_{e,n}^{\star i}} z^n$. We therefore obtain the following result.

Lemma 1. *For $n \ge 1$, let $\delta_{k,n} = t_{k,n-1} - t_{k,n}$. Then $t_{0,n} = 1/\mu$, $\delta_{0,n} = 0$ and, if $k \ge 1$,*

$$t_{k,n} = -\frac{1}{\mu} \sum_{i=1}^{k} \binom{k}{i} (-1)^i \overline{F_{e,n}^{\star i}}, \tag{2}$$

$$\delta_{k,n} = -\frac{1}{\mu} \sum_{i=1}^{k} \binom{k}{i} (-1)^i p_{e,n}^{*i}. \tag{3}$$

We now consider into detail the cases $k = 1, 2, 3$.

Lemma 2. *For $k \ge 2$, let $R_{k,n}^e = \overline{F_{e,n}^{\star k}} - k\overline{F}_{e,n}$ and $r_{k,n}^e = p_{e,n}^{*k} - kp_{e,n}$. Then*

$$t_{1,n} = \overline{F}_{e,n}/\mu \qquad \delta_{1,n} = p_{e,n}/\mu$$
$$t_{2,n} = -R_{2,n}^e/\mu \qquad \delta_{2,n} = -r_{2,n}^e/\mu$$
$$t_{3,n} = (R_{3,n}^e - 3R_{2,n}^e)/\mu \qquad \delta_{3,n} = (r_{3,n}^e - 3r_{2,n}^e)/\mu.$$

Proof. The results follow directly from (2) and (3). □

2.2. Asymptotic behaviour of $(t_{k,n})$ and $(\delta_{k,n})$, $1 \le k \le 3$.

In order to discuss the asymptotic behaviour of $(t_{k,n})$ and $(\delta_{k,n})$, we recall some basic definitions and properties of regularly varying sequences.

2.2.1. Regularly varying sequences

A sequence of real numbers (a_n) is regularly varying at infinity and with real index α if $a_n > 0$ for n large and if

$$\lim_{x\to\infty} \frac{a_{[xy]}}{a_{[x]}} = y^\alpha, \quad \forall y > 0. \tag{4}$$

Notation: $(a_n) \in RS(\alpha)$. We write $(a_n) \in RS$ if $(a_n) \in RS(\alpha)$ for some $\alpha \in \mathbb{R}$. If $(a_n) \in RS(\alpha)$, then (4) holds locally uniformly in $y > 0$; see Sections 1.2 and 1.9 in Bingham et al. (1987). From this it follows that $RS \subset LS$, where a sequence of real