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The Kumaraswamy skew-normal distribution

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ABSTRACT

We propose a new generalization of the skew-normal distribution (Azzalini, 1985) referred to as the Kumaraswamy skew-normal. The new distribution is computationally more tractable than the Beta skew-normal distribution (Mameli and Musio, 2013) with which it shares some properties.

1. Introduction

Some recent developments in distribution theory have proposed new techniques for building distributions. Among these, the methods used to construct the Beta generalized ($Beta - F$) (Jones, 2004) and the Kumaraswamy generalized ($Kw - F$) (Cordeiro and de Castro, 2011) class of distributions have received a lot of attention. The first work concerning the Beta-generated family was proposed by Eugene et al. (2002), who defined and analysed the Beta-normal distribution. Further, Jones (2004) formalized the definition of the Beta-generated family. Its work has inspired many researchers and has fuelled an enormous literature regarding this family of distributions; see for example Gupta and Nadarajah (2005), Pescim et al. (2010) and Mameli and Musio (2013). Recently, following the idea of the class of Beta-generated distributions (Jones, 2004), Cordeiro and de Castro (2011) proposed a new family of generalized distributions, called Kumaraswamy generalized family, by means of the Kumaraswamy distribution (Kumaraswamy, 1980; Jones, 2009). The maximum likelihood estimation for the family $Kw - F$ distribution results simpler than the estimation in the $Beta - F$ family. Motivated by these facts, we define in this paper a new generalization of the skew-normal based on the Kumaraswamy generalized family, which is more tractable of the Beta skew-normal (*BSN*) introduced by Mameli and Musio (2013). The resulting distribution, which will be called the Kumaraswamy skew-normal (*KwSN*), could be considered a valid alternative to the *BSN* distribution with which it shares some similar properties. Moreover, for special values of the parameters the *KwSN* distribution is related to the Beta skew-normal one. The *KwSN* distribution is always unimodal, unlike the Beta skew-normal which can be either unimodal or bimodal. The *KwSN* model shows more flexibility than the *SN* one. Furthermore, under the null hypothesis of normality the *KwSN* distribution, as the *BSN* one, is not identifiable. However, due to the tractability of the *KwSN* density, all the possible sets of parameters for which this density reduces to the normal one can be established by exploiting the Lambert W function; see Jeffrey et al. (1998). The rest of the paper organizes as follows. Section 2 defines the *KwSN* distribution and presents some properties of the new distribution. Section 3 investigates maximum likelihood estimation and analyses a data set of Australian athletes measurements. Finally, concluding remarks are given in Section 4.

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2. The new model

In this section we first define the Kumaraswamy skew-normal distribution and then we present some of its properties.

2.1. Definition and simple properties

For a given distribution function $F(x)$, with associated density function (pdf) $f(x)$, Cordeiro and de Castro (2011) represented the cumulative distribution function of the Kumaraswamy-F distribution as

$$G^K_{F(x)}(x; a, b) = 1 - (1 - F(x)^a)^b, \quad \text{with } x \in \mathbb{R}. \tag{1}$$

The density function correspondent to (1) is

$$g^K_{F(x)}(x; a, b) = abf(x)F(x)^{a-1}(1 - F(x)^a)^{b-1}, \quad \text{with } x \in \mathbb{R}. \tag{2}$$

The family of these distributions will be indicated by $Kw - F(a, b)$. Replacing the distribution function of the skew-normal $\Phi(x; \lambda)$ in (2), we obtain a new distribution, which will be called the Kumaraswamy skew-normal distribution ($KwSN(\lambda, a, b)$), whose density function is

$$g^K_{\Phi(x;\lambda)}(x; \lambda, a, b) = ab\phi(x; \lambda)(\Phi(x; \lambda))^{a-1}(1 - \Phi(x; \lambda)^a)^{b-1}, \quad \text{with } x \in \mathbb{R}. \tag{3}$$

This class can be generalized by including a location parameter μ and a scale parameter $\sigma > 0$. Thus if $X \sim KwSN(\lambda, a, b)$, then $Y = \mu + \sigma X$ is a $KwSN$ random variable with vector of parameters $\boldsymbol{\xi} = (\mu, \sigma, \lambda, a, b)$ or $Y \sim KwSN(\mu, \sigma, \lambda, a, b)$. Hereafter, we will denote by SN the skew-normal, by $N(0, 1)$ the normal, by KwN the Kumaraswamy-normal and by Kw the Kumaraswamy distributions.

We now give some simple properties of $KwSN(\lambda, a, b)$ density in (3).

Property 1. *Let $X \sim KwSN(\lambda, a, b)$.*

(a) *If $a = b = 1$, then $X \sim SN(\lambda)$.*
(b) *If $\lambda = 0$, then $X \sim KwN(a, b)$.*
(c) *If $a = b = 1$ and $\lambda = 0$, then $X \sim N(0, 1)$.*
(d) *If $a = \frac{1}{2}$, $b = 1$ and $\lambda = 1$, then $X \sim N(0, 1)$.*
(e) *If $a = 1$, $b = \frac{1}{2}$ and $\lambda = -1$, then $X \sim N(0, 1)$.*
(f) *Let $Y = \Phi(X; \lambda)$, then $Y \sim Kw(a, b)$.*
(g) *Let $Y = \Phi(X; \lambda)^a$, then $Y \sim Kw(1, b)$. Let $Z = 1 - Y$, then $Z \sim Kw(b, 1)$.*
(h) *As $\lambda \to +\infty$, the KwSN density tends to the Kumaraswamy half-normal density (Cordeiro et al., 2012).*
(i) *If $b = 1$, then $X \sim BSN(\lambda, a, 1)$.*

Proof. The results follow immediately by taking into account expression (3) and from elementary properties of the skew-normal distribution.

Graphical displays of the $KwSN$ density with various combinations of the parameters are shown in Fig. 1. It should be noted that the $KwSN$ density does not exhibit bimodality for any parameter values, unlike the BSN one which is unimodal or bimodal according to the region in which (λ, a, b) lie (Mameli and Musio, 2013).

As a result of the skewing mechanism introduced by Ferreira and Steel (2006), the $KwSN(\lambda, a, b)$ density can be represented as

$$g^K_{\Phi(x;\lambda)}(x; \lambda, a, b) = \phi(y)p(\Phi(y)), \quad -\infty < y < \infty$$

with $p(\cdot)$ on $(0, 1)$ given by

$$p(v; \lambda, a, b) = 2ab\Phi(\lambda\Phi^{-1}(v))\left(\Phi(\Phi^{-1}(v); \lambda)\right)^{a-1}\left(1 - \Phi(\Phi^{-1}(v); \lambda)^a\right)^{b-1}. \tag{4}$$

In view of this, the $KwSN(\lambda, a, b)$ density is a member of the unified skewed distributions of Abtahi et al. (2011). Therefore a stochastic representation of $X \sim KwSN(\lambda, a, b)$ follows from Proposition 1 in Abtahi et al. (2011). In particular, the following property is obtained.

Property 2. *Let U be a standard normal variable and let V be independent of U with pdf p on $(0, 1)$ given by Eq. (4).*

- *When $W = V - \Phi(U)$, the conditional distribution of U given $(W = 0)$ is $KwSN(\lambda, a, b)$.*
- *Let $X \sim KwSN(\lambda, a, b)$. Then $\Phi(X) \stackrel{d}{=} V$.*

Given the closed form of the cumulative distribution function, the $KwSN$ distribution can be easily simulated by using the probability integral transformation, or by using Property 1 item (f). Supposed that $a, b \geq 1$, another method for generating samples from the $KwSN$ is given by the algorithm of acceptance–rejection in Nadarajah et al. (2012).