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A dynamic view to moment matching of truncated distributions

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ABSTRACT

We derive an ordinary differential equation (ODE) for solving moment matching problems of truncated univariate distributions. Our method produces a trajectory that solves a family of moment matching problems for different truncation values all with the same target moments.

1. Introduction

The problem of matching distributional parameters to obtain desired moments is almost as old as the field of probability and statistics, see for example the classic book Shohat and Tamarkin (1943). In this paper we put forward an idea for solving the moment matching problem using a novel dynamic method, specifically suited for truncated distributions.

Consider a family of univariate continuous distributions (density functions), $\{g(x;\,\theta),\ \theta \in \Theta\}$, where the parameter space Θ is some subset of $\mathbb{R}^p$. Then, given desired moments, $m_1^*, \ldots, m_p^*$, the moment matching problem aims to find a solution, $\theta \in \Theta$, to the equations,

$$\int x^i g(x;\,\theta)\,dx = m_i^*, \quad i = 1, \ldots, p. \tag{1}$$

We consider the case where the density $g(\cdot\,;\,\theta)$ is a truncated distribution of the form,

$$g_{a,b}(x;\,\theta) = \frac{f(x;\,\theta)}{\int_a^b f(u;\,\theta)\,du},$$

where $\{f(x;\,\theta),\ \theta \in \Theta\}$ is a family of densities with support $\mathcal{S} \subset \mathbb{R}$ and $(a, b) \subset S$. Truncated distributions are ubiquitous in probability and statistics. See for example Robert (1995), Arismendi (2013) and Rosenbaum (1961).

We introduce a method to solve the moment matching problem for $g_{a,b}(\cdot\,;\,\cdot)$ by means of an ordinary differential equation (ODE). In fact, our method produces a range of solutions $\theta(z)$ with $z \in [0, 1]$, where for each z we have a solution to the moment matching problem (1) associated with $g_{a(z),b(z)}(\cdot\,;\,\cdot)$. Here $\big(a(0), b(0)\big) = \mathcal{S}$, $\big(a(1), b(1)\big) = (a, b)$ and as z

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increases from 0 to 1 the truncation interval $[a(z), b(z)]$ shrinks from $\mathcal{S}$ to the target truncation interval $[a, b]$. We use basic calculus to derive our ODE. The ODE for $\theta(z)$ describes the trajectory within the parameter space Θ such that the desired moments $m_1^*, \ldots, m_p^*$ are maintained throughout.

In practice, solving the moment matching problem when $[a, b] = \mathcal{S}$ is often explicit and immediate, but solving the truncated moment problem does not admit a closed form solution. As an example, consider the exponential distribution, $f(x;\, \theta) = \theta e^{-\theta x}\mathbf{1}\{x \in [0, \infty)\}$ where $\mathcal{S} = [0, \infty)$ with $\Theta = (0, \infty)$. In this case given a desired mean, m_1^*, the non-truncated moment matching solution is simply $\theta(0) = 1/m_1^*$, but for a truncation range $[a, b] \subset \mathcal{S}$ one needs to solve the equation (in θ),

$$m_1^* = \theta^{-1}\frac{(b\theta+1)e^{a\theta} - (a\theta+1)e^{b\theta}}{e^{a\theta} - e^{b\theta}}, \tag{2}$$

which does not admit a closed form solution. Thus to solve (2) one typically resorts to numerical methods, such as for e.g., Newton's method. Our dynamic (ODE based) approach is fundamentally different because it solves (2) for a whole range of truncation intervals simultaneously.

Besides the exponential distribution (which we handle mostly for the purpose of exposition), we also derive our equations for arbitrary continuous distributions with p parameters. We then specialise to location-scale families ($p = 2$), of which the normal distribution is a special case. Moment matching for truncated normal distributions has been previously studied in the literature, Dyer (1973) and Rosenbaum (1961), but to the best of our knowledge, the dynamic view which we put forward here is novel.

The remainder of the paper is structured as follows. Section 2 illustrates the main idea of the method through the exponential distribution. Section 3 puts forward general equations of our ODE based approach. Section 4 presents the solution for general location scale families and applies this to normal distributions. We conclude and pose open problems in Section 5.

2. Illustration of the main idea through the exponential distribution

To illustrate the main idea of our method, we consider the exponential distribution as presented in the introduction, but focus on the one-sided truncation $[0, b]$ with desired first moment, m_1^*. Note that by comparing to the (extreme case) of a uniform distribution, it is easy to see that it must hold that $m_1^* < b/2$.

Our goal is then to find a solution $\theta > 0$ for the equation,

$$\int_0^b x\frac{f(x;\, \theta)}{\int_0^b f(u;\, \theta)\, du}\, dx = m_1^*, \quad \text{or alternatively} \quad \int_0^b (x - m_1^*)f(x;\, \theta)\, dx = 0,$$

i.e., this is a solution to (2) (with $a = 0$).

We now take $z \in (0, 1]$ and for each z consider the truncation interval, $(0,\ b + (1-z)/z)$. For z close to zero, the interval is close to the non-truncated $[0, \infty)$ support and as z increases towards 1 we have that the interval shrinks to the target interval $[0, b]$. Now consider $\{\theta(z),\ z \in (0, 1]\}$ as a solution to the moment matching problem for every z. That is,

$$\int_0^{b+(1-z)/z} (x - m_1^*)f\big(x;\, \theta(z)\big)\, dx = 0, \quad z \in (0, 1].$$

Taking derivative with respect to z and using Leibniz's integral rule we get,

$$\int_0^{b+(1-z)/z} (x - m_1^*)\frac{d}{dz}f\big(x;\, \theta(z)\big)\, dx - \frac{1}{z^2}\left(b + \frac{1-z}{z} - m_1^*\right)f\left(b + \frac{1-z}{z}\,;\, \theta(z)\right) = 0. \tag{3}$$

Now observe that,

$$\frac{d}{dz}f\big(x;\, \theta(z)\big) = \theta'(z)e^{-\theta(z)x}(1 - \theta(z)x),$$

and plug this derivative into (3). After rearranging, explicitly carrying out the integration and further simplifying we get the ODE:

$$\begin{aligned}\theta'(z) &= \frac{\frac{1}{z^2}(b + \frac{1-z}{z} - m_1^*)\theta(z)e^{-\theta(z)\left(b+\frac{1-z}{z}\right)}}{\int_0^{b+(1-z)/z}(x - m_1^*)e^{-\theta(z)x}(1 - \theta(z)x)\, dx} \\ &= \theta^3(z)\frac{b + \frac{1-z}{z} - m_1^*}{\big((b-1)z+1\big)\big(z\theta(z)(b-m-1) + z + \theta(z)\big)\theta(z) + z^2(1 - e^{\theta(z)(b+\frac{1-z}{z})})}.\end{aligned} \tag{4}$$

As an initial condition for the ODE, we take $\theta(0^+) = \frac{1}{m_1^*}$ since this is the solution of the non-truncated moment matching problem.