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A new kind of augmentation of filtrations suitable for a change of probability measure by a strict local martingale[☆]

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ABSTRACT

In this note we introduce a new kind of augmentation of filtrations along a sequence of stopping times. This augmentation is suitable for the construction of new probability measures associated to a positive strict local martingale as done in Kardaras et al. (2015), while it is on the other hand rich enough to make classical results from stochastic analysis hold true on some stochastic interval of interest.

1. Introduction

The goal of this paper is to introduce a new kind of augmentation of filtrations which is suitable for a change of probability measure associated with a strict local martingale. While it is safe and very convenient to work under the usual assumptions when doing a change of probability measure where the density process is a uniformly integrable martingale, one must be more careful if one takes a non-uniformly integrable martingale or a strict local martingale as a "potential" Radon–Nikodym density process.

Indeed it was already noted in Bichteler's book (2002) and later by Najnudel and Nikeghbali (2011) that in order to extend a consistent family of probability measures from $\bigcup_{t\geq 0}\mathcal{F}_t$ to $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0}\mathcal{F}_t\right)$ one has to impose certain topological requirements on the probability space and one must refrain from the usual assumptions. This is however rather unsatisfactory in general, since results from stochastic analysis like the existence of regular versions of martingales do require some augmentation of the filtration. The existence of such versions is for example of interest whenever one considers an uncountable number of stochastic processes as it is often the case in dynamic optimization problems. This led Najnudel and Nikeghbali (2011) to introduce a new kind of augmentation of filtrations, the natural augmentation, that is compatible with the construction of a probability measure on $\mathcal{F}_\infty$ whose density process is defined via a non-uniformly integrable martingale.

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While a positive strict local martingale (X_t), i.e. a local martingale which is not a true martingale, cannot directly serve as a Radon–Nikodym density process, it is still possible to construct a new measure Q on $\mathcal{F}_{\tau^X -}$ by extending the consistent family of measures Q_n defined on $\mathcal{F}_{\tau_n^X}$ by

$$\mathsf{Q}_n = X_{\tau_n^X} \cdot \mathsf{P}, \quad \tau_n^X = \inf\{t \geq 0 : X_t > n\} \wedge n, \ \tau^X = \lim_{n\to\infty} \tau_n^X,$$

if the filtration on the underlying probability space is the right-augmentation of a so called standard system. Standard systems were introduced by Parthasarathy (1967) and first used in the above context by Föllmer (1972). Since in this case the measure Q is only uniquely defined on the sub-σ-algebra $\mathcal{F}_{\tau^X -}$ and is generally not absolutely continuous with respect to P on $\mathcal{F}_t$ for all $t \in \mathbb{R}$, we cannot use the natural augmentation introduced by Najnudel and Nikeghbali (2011). While the problem in Najnudel and Nikeghbali (2011) was the inclusion of null-sets from $\mathcal{F}_\infty$ in the initial filtration $\mathcal{F}_0$, the problem now becomes even more severe in that one can no longer include any null-events that happen after time τ^X in the initial filtration $\mathcal{F}_0$. This observation leads us to introduce a new kind of augmentation of filtrations along a sequence of stopping times that is on the one hand rich enough to make classical results from stochastic analysis hold true up to some stopping time and that on the other hand still allows for the construction of the new probability measure.

This paper is organized as follows: in the next section we introduce a new kind of augmentation of filtrations along an increasing sequence of stopping times and we establish the existence of nice versions of stochastic processes up to some stopping time under the new augmentation. In Section 3 we briefly review the construction of the above mentioned probability measure associated with a positive (strict) local martingale, before we apply the augmentation results from Section 2 in this setting.

2. (τ_n)-natural assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space. We will start with a definition before stating the augmentation theorem.

Definition 2.1. Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of $(\mathcal{F}_t)$-stopping times.

- A subset $A \in \Omega$ is called $(\tau_n)_{n\in\mathbb{N}}$-negligible with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, iff there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of subsets of Ω s.t. for all $n \in \mathbb{N}$, $B_n \in \mathcal{F}_{\tau_n}$, $\mathsf{P}(B_n) = 0$, and $A \subset \bigcup_{n\in\mathbb{N}} B_n$.
- We say that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ is (τ_n)-complete, iff all the (τ_n)-negligible sets of Ω are contained in $\mathcal{F}_0$. It satisfies the (τ_n)-natural conditions, iff it is (τ_n)-complete and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous.

Note that in the case of $\tau_n = n$, the above definition as well as the next theorem reduces to the case of the natural augmentation studied in Bichteler (2002) and Najnudel and Nikeghbali (2011), where all $\mathcal{F}_n^+$-negligible sets for all $n \in \mathbb{N}$ are included in $\mathcal{F}_0$. Thus, the following theorem can be seen as a generalization of Proposition 2.4 in Najnudel and Nikeghbali (2011).

Theorem 2.2. *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of stopping times on a filtered probability $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ and denote by $\mathcal{N}$ the family of all (τ_n)-negligible sets with respect to* P*. Set $\widetilde{\mathcal{F}} = \sigma(\mathcal{F}, \mathcal{N})$ and $\widetilde{\mathcal{F}}_t = \sigma(\mathcal{F}_t^+, \mathcal{N})$ for all $t \geq 0$. Then there exists a unique probability measure $\widetilde{\mathsf{P}}$ on $(\Omega, \widetilde{\mathcal{F}})$, which coincides with* P *on $\mathcal{F}$, and the space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ satisfies the (τ_n)-natural conditions. Moreover, $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ is the smallest extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, which satisfies the (τ_n)-natural conditions. We therefore call it the (τ_n)-augmentation of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$.*

Proof. We only give a sketch of the proof, because all steps except the third one (which we do in detail) follow closely the proof of Proposition 2.4 in Najnudel and Nikeghbali (2011).

(1) Define $\mathcal{E} = \{A \subset \Omega \,|\, \exists A' \in \mathcal{F} : A \Delta A' \in \mathcal{N}\}$. As in Najnudel and Nikeghbali (2011) it is easily checked that $\mathcal{E}$ is a σ-algebra and that $\mathcal{E} = \widetilde{\mathcal{F}}$. This implies that if $\widetilde{\mathsf{P}}$ is a probability on $(\Omega, \widetilde{\mathcal{F}})$ extending P we must have $\widetilde{\mathsf{P}}(A) = \mathsf{P}(A')$ for $A \in \widetilde{\mathcal{F}}$, where $A' \in \mathcal{F}$ satisfies $A \Delta A' \in \mathcal{N}$. Therefore, the measure $\widetilde{\mathsf{P}}$ is unique if it exists. Furthermore, $\widetilde{\mathcal{F}}_t = \{A \subset \Omega \,|\, \exists A' \in \mathcal{F}_t^+ : A\Delta A' \in \mathcal{N}\}$ for all $t \geq 0$ as can be easily verified.

(2) Next we show that $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ is right-continuous:
For this assume that $A \in \bigcap_{s>t} \widetilde{\mathcal{F}}_s$. Therefore, $A \in \widetilde{\mathcal{F}}_{t+1/n}$ for all $n \in \mathbb{N}$ and there exists $A'_n \in \mathcal{F}_{(t+1/n)+}$ such that $A\Delta A'_n \in \mathcal{N}$ for all $n \in \mathbb{N}$. But then

$$A' := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A'_n \in \mathcal{F}_{t+}$$

and

$$A\Delta A' = A\Delta \left[\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A'_n \right] \in \mathcal{N},$$

which implies that $A \in \widetilde{\mathcal{F}}_t$.