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Small deviation probabilities of weighted sums under minimal moment assumptions[☆]

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ABSTRACT

We examine small deviation probabilities of weighted sums of i.i.d.r.v. with a power decay at zero under moment assumptions close to necessary.

1. Introduction

Let $\{X_n\}_{n\geq 1}$ be independent copies of a positive random variable X with a distribution function $F(x) = \mathbf{P}(X < x)$ and let λ be a bounded positive non-increasing (without loss of generality) function defined on the interval $[1, \infty]$ such that the series $S = \sum_{n\geq 1} \lambda(n) X_n$ converges almost surely (a.s.), or by Kolmogorov's Three-Series Theorem equivalently

$$\sum_{n\geq 1} \mathbf{E} \min(1, \lambda(n)X) < \infty \tag{1.1}$$

(see (3.1) in Rozovsky, 2007 or (0.2) in Borovkov and Ruzankin, 2008).

Remind that the last condition lays certain moment restrictions on X. So, if $\lambda(n) = n^{-A},\ A > 1$, then (1.1) is equivalent to the condition $\mathbf{E}X^{1/A} < \infty$, and if $\lambda(n) = q^n,\ 0 < q < 1$, then (1.1) is equivalent to $\mathbf{E}\log(1 + X) < \infty$.

The subject of our interest is the small deviation probabilities of the series S, i.e. the asymptotic behavior of $\mathbf{P}(S < r)$ as $r \to 0$. Let us now formulate some conclusions from Dunker et al. (1998), Lifshits (1997) and Rozovsky (2007), which are tightly connected with the obtained results. Note that the history of the question and the extensive list of references by the theme one can find in Lifshits (1997), and also that the exhaustive bibliography on small deviation probabilities contains in Lifshits (2012).

Theorem 1, a refinement of the original result from Lifshits (1997) (see also Dunker et al., 1998, Theorem 2.2), was proved in Rozovsky (2007). To formulate it we have to introduce two conditions:

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Condition **L**. There exist constants $b \in (0, 1)$, $c_1, c_2 > 1$ and $\varepsilon > 0$ such that for each $r \le \varepsilon$ the relation $c_1 F(br) \le F(r) \le c_2 F(br)$ holds.

Condition **F**. $\limsup_{s\to\infty} s^2 (1 - F(s))/\mathbf{E}X^2\, \mathbf{1}[X < s] < \infty$.

Pay attention that condition **L**, for the first time offered in Lifshits (1997), is an essential relaxation of the assumption $F(1/\cdot) \in \mathbf{R}_\alpha$, the class of regularly varying functions of order $\alpha < 0$.

Concerning condition **F** (the so-called Feller stochastic compactness of the distribution function of F), we mention that it is rather mild (for instance, **F** holds if X belongs to the domain of attraction of any stable law). Nevertheless, **F** implies $\mathbf{E}X^\delta < \infty$ for some positive δ (see Rozovsky, 2007, (2.2)).

Set

$$f(u) = \mathbf{E}e^{-uX}, \qquad L(u) = \sum_{n\ge 1} \log f(u\lambda(n)), \quad u \ge 0.$$

Theorem 1. *Let (1.1) hold and the distribution function F satisfy Conditions* **L** *and* **F**. *Then we have as $r \to 0^+$*

$$\mathbf{P}(S < r) \sim \frac{\exp(L(u) + ur)}{\sqrt{2\pi u^2 L''(u)}}, \tag{1.2}$$

where $u = u(r)$ is any function tending to $+\infty$ so that

$$\frac{uL'(u) + ur}{\sqrt{u^2 L''(u)}} = o(1). \tag{1.3}$$

We remark that the standard choice of u in (1.3) is a unique solution of

$$L'(u) + r = 0. \tag{1.4}$$

However, in applications it is very convenient to use approximate solutions of (1.4).

The next result is an improvement of Theorem 3.1 from Dunker et al. (1998) (in particular, by using Rozovsky, 2007 instead of Lifshits, 1997, we succeeded to relax the moment assumptions of the mentioned theorem from $\mathbf{E}X^2 < \infty$ up to (1.1) and **F**).

Introduce the notation:

$$I_0(u) = \int_1^\infty \log f(u\lambda(t))\, dt,$$
$$I_1(u) = \int_1^\infty u\lambda(t)\, (\log f)'(u\lambda(t))\, dt,$$
$$I_2(u) = \int_1^\infty (u\lambda(t))^2\, (\log f)''(u\lambda(t))\, dt.$$

Theorem 2. *Let $\lambda(\cdot)$ (see (1.1)) be a logarithmically convex, twice differentiable and integrable function on $[1, \infty]$. Assume that (1.1) and* **F** *hold and, moreover, distribution function $F(1/\cdot) \in \mathbf{R}_\alpha$ for some $\alpha < 0$ and possesses the property:*

Condition **I**. *The function $(s\, (\log f)'(s))'$ is absolutely integrable at infinity.*

Then as $r \to 0^+$

$$\mathbf{P}(S < r) \sim \sqrt{\frac{\Gamma(1-\alpha) F(1/u\lambda(1))}{2\pi I_2(u)}}\, \exp(I_0(u) + \rho(u) + ur), \tag{1.5}$$

where the explicitly defined function ρ is bounded and $u = u(r)$ is any function tending to $+\infty$ so that

$$\frac{I_1(u) + ur}{\sqrt{I_2(u)}} = o(1). \tag{1.6}$$

Mention that several conditions under which **I** holds were given in Dunker et al. (1998) and Rozovsky (2012). For instance, one can assume that the function $u\,\big(\log F(u)\big)'$ tends monotonically to $-\alpha$ as $u \searrow 0$. Note that in this situation $F(1/\cdot) \in \mathbf{R}_\alpha$.

We see that the assertions of Theorems 1 and 2 are heavily dependent on condition **F** which can be rather far from the necessary condition (1.1), especially if $\lambda(n)$ tends to zero fast enough as in the case $\lambda(n) = q^n$, $0 < q < 1$, say.

Thus, it is of interest to clear as the small deviations of S behave if **F** is violated.

The basic purpose of the present note is to find the asymptotic expression for $\mathbf{P}(S < r)$ under condition **L** provided that the moment assumptions are close to optimal.