Statistics and Probability Letters 86 (2014) 50–60

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Quasi-maximum likelihood estimation for multiple volatility shifts

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A R T I C L E I N F O

Article history:
Received 7 August 2013
Received in revised form 26 November 2013
Accepted 7 December 2013
Available online 25 December 2013

MSC:
primary 62M10
62G10
secondary 60G18

Keywords:
Volatility shifts
Change point analysis
Quasi-maximum likelihood estimation

A B S T R A C T

We propose the Gaussian quasi-maximum likelihood estimator (QMLE) to detect and locate multiple volatility shifts. Our Gaussian QMLE is shown to be consistent under suitable conditions and the rate of convergence is provided. It is also shown that the binary segmentation procedure provides a consistent estimation for the number of volatility shifts.

1. Introduction

In the analysis of time-varying volatility of financial time series, high persistence in volatility has been widely recognized as one of the most prominent features. To capture such high persistence, the integrated GARCH models were once introduced by Engle and Bollerslev (1986). However, some authors argued that high persistence may occur due to structural breaks in volatility. For relevant references, we refer to Lamoureux and Lastrapes (1990), Mikosch and Stărică (2004) and Hillebrand (2005). Besides, it has been also revealed that the existence of volatility shifts can result in spurious long range dependence in volatility (see, for example, Klemeš (1974), Teverovsky et al. (1999), Diebold and Inoue (2001), Mikosch and Stărică (2004) and references therein). Accordingly, it has been in agreement that structural breaks in volatility should be taken into consideration when modeling the volatility of financial time series.

The problem of detecting the change points, or structural breaks has been studied for decades in the context of changes in mean. Among many others, we will refer to Csörgő and Horváth (1997), Perron (2006), Aue and Horváth (2013) and references therein for comprehensive review. Recently, with a remarkable attention to volatility shifts of financial time series, detecting and locating structural changes in volatility has attracted many researchers. For example, Kokoszka and Leipus (2000) suggested a CUSUM type change-point estimator in ARCH models with a single volatility shift. Andreou

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http://dx.doi.org/10.1016/j.spl.2013.12.007

and Ghysels (2002) conducted an extensive simulation study to compare the performance of existing methods for various multiple volatility shifts. Davis et al. (2008) considered multiple breaks detection for a class of segmented GARCH processes based on the minimum description length algorithm.

This paper introduces the Gaussian quasi-maximum likelihood estimator (QMLE) to find the location of multiple volatility shifts and the binary segmentation procedure to estimate the number of volatility shifts. Our proposed method is very straightforward and easy to implement. From a theoretical viewpoint, our Gaussian QMLE is shown to be consistent and converges in an order of sample size, namely n-consistent. It is also shown that the binary segmentation procedure provides a consistent estimation for the number of volatility shifts. Monte Carlo simulations support that our proposed method performs reasonably well even for non-Gaussian settings.

2. Estimating the locations of multiple volatility shifts

We consider the following volatility shifts model. For a set of break fractions $0 =: \lambda_0 < \lambda_1 < \cdots < \lambda_R < \lambda_{R+1} := 1$, let

$$r_t = r_{i,t}, \quad \text{if } [n\lambda_i] < t \le [n\lambda_{i+1}], \qquad \sigma_i^2 = \mathrm{E}r_{i,1}^2 < \infty, \quad \text{for } i = 0, \dots, R, \tag{2.1}$$

where $\{(r_{0,t}, \dots, r_{R,t}) : t \in \mathbb{Z}\}$ is strictly stationary and ergodic with mean zero. It is assumed that each σ_i^2 is strictly positive and $\sigma_{i-1}^2 \ne \sigma_i^2$ for every $i = 1, \dots, R$. In financial applications, $\{r_t\}$ usually represents a series of log-returns with R volatility shifts, and each $[n\lambda_i]$ denotes the ith break point. Furthermore, we will write $\lambda_1^\circ, \dots, \lambda_R^\circ$ as the true locations of multiple volatility shifts satisfying

$$\lambda_{i+1}^\circ - \lambda_i^\circ > \varsigma_0 \quad \text{for each } i = 0, \dots, R,$$

for a sufficiently small number $\varsigma_0 > 0$. The true volatility on an interval $([n\lambda_i^\circ], [n\lambda_{i+1}^\circ]]$ is denoted by $\mathrm{v}_i := \mathrm{E}r_{i,0}^2$, $i = 0, \dots, R$. In practice, we observe $\{r_1, \dots, r_n\}$ with *unknown* number of volatility shifts. Here, we will consider the estimation of $\lambda_1^\circ, \dots, \lambda_R^\circ$ when R is *known*. Statistical inference on the *unknown* number of volatility shifts R will be discussed in Section 3.

The idea behind our proposed estimator is very straightforward. Suppose that $\{r_t\}$ in (2.1) are independent observations from Normal distribution. Then, the negative log-likelihood is given by

$$\mathcal{L}(\lambda_1, \dots, \lambda_R, \sigma_0^2, \dots, \sigma_R^2) = \sum_{i=0}^{R} \sum_{t=[n\lambda_i]+1}^{[n\lambda_{i+1}]} \left\{ \frac{r_t^2}{\sigma_i^2} + \log \sigma_i^2 \right\}. \tag{2.2}$$

By minimizing $\mathcal{L}$ with respect to nuisance parameters $\sigma_0^2, \dots, \sigma_R^2$, that is, solving $\partial\mathcal{L}/\partial\sigma_i^2 = 0, i = 0, \dots, R$, we obtain that

$$\widehat{\sigma}_i^2 = \frac{1}{[n\lambda_{i+1}] - [n\lambda_i]} \sum_{t=[n\lambda_i]+1}^{[n\lambda_{i+1}]} r_t^2. \tag{2.3}$$

By plug-in (2.3) to (2.2), a profile likelihood becomes

$$\mathcal{L}(\lambda_1, \dots, \lambda_R, \widehat{\sigma}_0^2, \dots, \widehat{\sigma}_R^2) = \sum_{i=0}^{R} ([n\lambda_{i+1}] - [n\lambda_i]) \left\{ 1 + \log \frac{1}{[n\lambda_{i+1}] - [n\lambda_i]} \sum_{t=[n\lambda_i]+1}^{[n\lambda_{i+1}]} r_t^2 \right\}. \tag{2.4}$$

Therefore, we obtain the Gaussian MLE by minimizing $\mathcal{L}$ with respect to $(\lambda_1, \dots, \lambda_R)$. For dependent observations $\{r_t\}$, (2.2) divided by sample size n can be interpreted as the method of moment estimator of

$$\sum_{i=0}^{R} \left\{ \frac{1}{\sigma_i^2} \int_{\lambda_i}^{\lambda_{i+1}} z(\lambda) d\lambda + (\lambda_{i+1} - \lambda_i) \log \sigma_i^2 \right\}, \quad z(\lambda) = \sum_{i=0}^{R} \mathrm{v}_i I\,(\lambda_i^\circ < \lambda \le \lambda_{i+1}^\circ).$$

Since the counterpart is minimized at $(\lambda_1, \dots, \lambda_R) = (\lambda_1^\circ, \dots, \lambda_R^\circ)$ and $(\sigma_0^2, \dots, \sigma_R^2) = (\mathrm{v}_0, \dots, \mathrm{v}_R)$ (cf. the proof of Theorem 1), the estimator minimizing $\mathcal{L}$ is reasonable.

Finally, we define our Gaussian QMLE as

$$(\hat{\lambda}_1, \dots, \hat{\lambda}_R) := \operatorname{argmin} \{\mathrm{M}_n(\lambda_1, \dots, \lambda_R) : \boldsymbol{\lambda} \in \mathrm{D}_R\}, \tag{2.5}$$

where

$$\mathrm{M}_n(\boldsymbol{\lambda}) := \sum_{i=0}^{R} \mathrm{L}_n(\lambda_i, \lambda_{i+1}),$$

$$\mathrm{L}_n(\lambda_i, \lambda_{i+1}) := (\lambda_{i+1} - \lambda_i) \left\{ 1 + \log \frac{1}{n(\lambda_{i+1} - \lambda_i)} \sum_{t=[n\lambda_i]+1}^{[n\lambda_{i+1}]} r_t^2 \right\}$$