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Convergence to the maximum process of a fractional Brownian motion with shot noise

Yizao Wang

Department of Mathematical Sciences, University of Cincinnati, 2815 Commons Way, ML-0025, Cincinnati, OH, 45221-0025, United States

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ABSTRACT

We consider the maximum process of a random walk with additive independent noise in the form of $\max_{i=1,\ldots,n}(S_i + Y_i)$. The random walk may have dependent increments, but its sample path is assumed to converge weakly to a fractional Brownian motion. When the largest noise has the same order as the maximal displacement of the random walk, we establish an invariance principle for the maximum process in the Skorohod topology. The limiting process is the maximum process of the fractional Brownian notion with shot noise generated by Poisson point processes.

1. Introduction

Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two independent sequences of random variables. Write $S_0 = 0, S_n = X_1 + \cdots + X_n$ and we are interested in the asymptotic behavior of the maximum process $M_0 = 0$,

$$M_n = \max_{i=1,\ldots,n}(S_i + Y_i), \quad n \in \mathbb{N}.$$

We view $\{S_n\}_{n\in\mathbb{N}}$ as a random walk and $\{Y_n\}_{n\in\mathbb{N}}$ perturbations (or noise). We allow dependence between steps of random walk, while the perturbations $\{Y_n\}_{n\in\mathbb{N}}$ are assumed to be independent and identically distributed (i.i.d.).

We consider the general framework that the sample path of the random walk $\{S_{\lfloor nt\rfloor}\}_{t\in[0,1]}$ without perturbation converges weakly to a stochastic process, after appropriate normalization. Such results are referred to as invariance principles (and/or functional central limit theorems) in the literature. This includes Donsker's theorem (Donsker, 1951) which states that when $\{X_n\}_{n\in\mathbb{N}}$ are i.i.d. with zero mean and unit variance, the sample path converges weakly to a standard Brownian motion.

More generally when $\{X_n\}_{n\in\mathbb{N}}$ is stationary, weak convergence to fractional Brownian motions has been extensively investigated. Throughout, we assume the following invariance principle to hold for S_n:

$$\left\{\frac{S_{\lfloor tn\rfloor}}{n^H}\right\}_{t\in[0,1]} \Rightarrow \left\{\mathbb{B}_t^H\right\}_{t\in[0,1]} \tag{1}$$

E-mail addresses: yizao.wang@uc.edu, yizwang@umich.edu.

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in the space of càdlàg functions $D[0,1]$, where $\mathbb{B}^H = \{\mathbb{B}_t^H\}_{t\in[0,1]}$ is the fractional Brownian motion with Hurst index $H \in (0,1)$. This is the zero mean Gaussian process with covariance function $\mathrm{Cov}(\mathbb{B}_s, \mathbb{B}_t) = 2^{-1}(s^{2H} + t^{2H} + |s-t|^{2H})$, $s, t \geq 0$. An extensively studied model of $\{X_n\}_{n\in\mathbb{N}}$ leading to the invariance principle (1) is the stationary linear process. The seminal work of Taqqu (1975) addressed the case when innovations of the linear processes are independent; for dependent innovations, we refer to the recent result of Dedecker et al. (2011) and references therein, among many others under various dependence assumptions.

We are interested in the behavior of the maximum process of $\{S_n\}_{n\in\mathbb{N}}$ perturbed with $\{Y_n\}_{n\in\mathbb{N}}$, given that the invariance principle (1) holds for some $H \in (0,1)$. It is clear that the only non-trivial case is when the tail distribution of Y_i satisfies, for some $\kappa > 0$,

$$\mathbb{P}(Y_i > x) \sim \kappa x^{-1/H} \quad \text{as } x \to \infty. \tag{2}$$

Indeed, in this case,

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\dots,n} \frac{Y_i}{n^H} \leq x\right) = \exp(-\kappa x^{-1/H}), \quad x > 0,$$

and the normalization n^H gives the non-degenerate distributional limit for both S_n and Y_n. Otherwise, if $\mathbb{P}(Y_i > x) \sim \kappa x^{-\beta}$ for some $\beta \neq 1/H$, then either $\max_{i=1,\dots,n} S_i$ or $\max_{i=1,\dots,n} Y_i$ will dominate after appropriate normalization, and the result is immediate.

The main result of this paper is an invariance principle in the form of

$$\left\{\frac{M_{\lfloor nt\rfloor}}{n^H}\right\}_{t\in[0,1]} \Rightarrow \{Z_t^H\}_{t\in[0,1]}, \quad \text{in } D[0,1]. \tag{3}$$

The limiting process $\{Z_t^H\}_{t\in\mathbb{R}_+}$ is defined as follows. Let $\mathbb{B}^H = \{\mathbb{B}_t^H\}_{t\in\mathbb{R}_+}$ be a fractional Brownian motion, and let $\{\eta_i, U_i\}_{i\in\mathbb{N}}$ be a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $H^{-1}x^{-1-1/H}dxdu$, in the same probability space as and independent of $\{\mathbb{B}_t^H\}_{t\in\mathbb{R}_+}$. Throughout, we assume $\mathbb{B}^H$ has continuous sample path (Samorodnitsky and Taqqu, 1994). Then, the limiting process is defined as

$$Z_t^H(\kappa) := \sup_{s\in[0,t]}\left(\mathbb{B}_s^H + \kappa^H \sum_{i=1}^\infty \eta_i \mathbf{1}_{\{U_i=s\}}\right), \quad t \in \mathbb{R}_+. \tag{4}$$

In the sequel, we fix $\kappa \in (0,\infty)$, $H \in (0,1)$ and write $Z^H \equiv \{Z_t^H\}_{t\in\mathbb{R}_+} \equiv \{Z_t^H(\kappa)\}_{t\in\mathbb{R}_+}$ for the sake of simplicity. The main result is the following.

Theorem 1. *Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two independent sequences of random variables. Suppose that $\{X_n\}_{n\in\mathbb{N}}$ satisfy (1) and $\{Y_n\}_{n\in\mathbb{N}}$ are i.i.d. random variables satisfying (2). Then, the invariance principle (3) holds in the space of $D[0,1]$ equipped with the Skorohod J-1 topology.*

We refer to Z^H in (4) as the maximum process of *fractional Brownian motion with shot noise*. To the best of our knowledge, the limiting process Z^H is new.

Our motivation originally came from Hitczenko and Wesołowski (2011) in the study of perpetuities, who raised an open question in our framework with S_n converging weakly to a standard Brownian motion (the open question was actually on the marginal distribution of M_n). Theorem 1 indicates that the conjectured limiting object in Hitczenko and Wesołowski (2011, Remarks, p. 889) is incorrect.

The model $\{S_n + Y_n\}_{n\in\mathbb{N}}$ has been seen in the literature as the *perturbed random walk*, see for example Araman and Glynn (2006) and Alsmeyer et al. (2014), and references therein for other motivations and applications. Here we take a different aspect, while most of the results in the literature assume negative drift of the random walk $\{S_n\}_{n\in\mathbb{N}}$. Moreover, the same name *perturbed random walk* has been used for another model of self-interacting random walk, see Davis (1996) and Perman and Werner (1997). The corresponding limiting process, the so-called *perturbed Brownian motion*, has also been characterized and investigated. We choose not to use the name *perturbed fractional Brownian motion* for our process Z^H.

Theorem 1 is first proved in the case when $\{Y_n\}_{n\in\mathbb{N}}$ are non-negative. For this part the proof is essentially an application of the continuous mapping theorem combined with a truncation argument. To do so, recall the invariance principle for S_n (1) and also the weak convergence for order statistics of $\{Y_n\}_{n\in\mathbb{N}}$. Indeed, under (2), the order statistics $Y_{1,n} \geq Y_{2,n} \geq \cdots \geq Y_{n,n}$ of $Y_1, \dots, Y_n$ satisfy

$$\left\{\left(\frac{Y_{i,n}}{n^H}, \frac{i}{n}\right)\right\}_{i=1,\dots,n} \Rightarrow \left\{(\kappa^H \eta_n, U_n)\right\}_{n\in\mathbb{N}} \tag{5}$$

in the space of Radon point measures on $(0,\infty) \times (0,1)$, where $(\eta, \mathbf{U}) = \{(\eta_n, U_n)\}_{n\in\mathbb{N}}$ as before is a Poisson point process on $\mathbb{R}_+ \times (0,1)$ with intensity measure $H^{-1}x^{-1-1/H}dxdu$. See the seminal work of LePage et al. (1981) and also Resnick (1987, Corollary 4.19). If one could represent Z^H as the image of a continuous function evaluated at $\mathbb{B}^H$ and $(\eta, \mathbf{U})$, then the result would be immediate. In the proof, however, a continuous mapping is constructed for truncated versions of Z^H, and we