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On piecewise linear processes

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ABSTRACT

The letter concerns piecewise deterministic processes controlled by a Markov flow with exponentially, $\mathrm{Exp}(\lambda_n)$, distributed interarrival times T_n. Assuming all rates λ_n to be different, we study the distribution of a piecewise linear process with jumps.

1. Introduction

Let $T_n, \ n \in \mathbb{N}$, be independent exponentially distributed (with rates $\lambda_n, \ \lambda_n > 0$) random variables defined on some probability space:

$$\mathbb{P}\{T_n > t\} = \mathrm{e}^{-\lambda_n t}, \quad t > 0.$$

Consider a particle which moves on the line with constant velocities $c_n, \ n \geq 1$. It starts from the origin with velocity c_1, afterwards it changes the velocities at the random times. The particle moves with velocity c_n during random time T_n, so the switchings occur at the times $T^{(n)} = T_1 + \cdots + T_n, \ n \in \mathbb{N}$. The position $X(t)$ of the particle at time $t, \ t \geq 0$, is described by the random process with piecewise linear sample paths. Such behaviour corresponds to the so-called piecewise-deterministic Markov processes, which were first defined in Gnedenko and Kovalenko (1966) (see also Davis (1984)). Subsequently, these models have been generalised up to the processes, which take values in a general Borel space, see the review by Costa and Dufour (2013) (see also Jacobsen (2006, Chapter 7)).

The renewal approach to such processes have been developed by Cox (1962), where the Laplace transformation methods have been exploited. In particular, various formulae for the distributions of $T^{(n)}$ and of the counting processes can be found there, see below (2.1), (2.4) (Section 2), and formulae (1.4.3)–(1.4.4) in Cox (1962). Here, by using different methods, we compute also the joint distribution of $(T_1, \ldots, T_n)\mathbb{1}_{\{N(t)=n\}}$ and the density function $\pi^{(n)}(\cdot, t)$ of r.v. $T^{(n)}\mathbb{1}_{\{N(t)=n\}}, \ n \in \mathbb{N}, \ t > 0$, where $N(t)$ is the counting Poisson process.

The piecewise linear processes with alternating velocities c_n and with exponentially distributed inter-switching times (the so-called telegraph processes) are the most studied, see the review Kolesnik and Ratanov (2013).

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The case of exponentially distributed inter-switching times (with the special dependence between the switching intensities) is studied in detail by Di Crescenzo and Martinucci (2010) and Di Crescenzo et al. (2012) in terms of the k-fold convolutions of inter-switching times.

Nevertheless, to the best of my knowledge, a detailed analysis of general 1D piecewise-linear processes (with different velocities and intensities of switchings) is not presented in the literature. This letter fills the gap.

The piecewise-deterministic Markov processes originally have been designed for queueing theory, see Gnedenko and Kovalenko (1966) now they are used for modelling in various fields, see e.g. Davis (1984, 1993), Costa and Davis (1989), de Saporta and Dufour (2012), Costa and Dufour (2013). These applications also include financial modelling, see the seminal paper Cox and Ross (1975), where the complete market model based on a pure jump process has been constructed. A more detailed modern version of this model is presented in Jacobsen (2006, Chapter 10). *In both of these cases all the velocities c_n are assumed to be equal.* The case with alternating velocities is studied in Ratanov (2007). This model exploits telegraph processes with *rates λ and slopes c*, *alternating at random times*, and *with jumps occurring at the times of velocity reversals* (the so-called jump-telegraph processes), see also the review in Kolesnik and Ratanov (2013).

In this paper, assuming that all rates λ are different, we study the distribution of a piecewise constant process, see Section 2, and then, of a piecewise linear process. Being motivated by possible applications to financial modelling, in Section 3 we derive the formulae for the moment generating function (Theorem 3.1) and for the expectation of the piecewise linear process provided with a pure jump component (Theorem 3.2). This process becomes a martingale under certain conditions similar to the case of the jump-telegraph processes (see Kolesnik and Ratanov (2013)). We derive also the explicit distribution of the piecewise linear process (with jumps) for the special case of alternating velocities, $c_n = c(-1)^{n+1}$ (Theorem 3.3).

We will repeatedly use the following notations. Consider the functions

$$\Phi_n(t) = \frac{1}{\lambda_{n+1}} \sum_{j=1}^{n+1} A_{n+1,j}\lambda_j \mathrm{e}^{-\lambda_j t}, \quad t \ge 0,\ n = 0, 1, 2, 3, \dots, \tag{1.1}$$

where

$$A_{n,j} = \prod_{\substack{1\le k\le n,\\ k\ne j}} \frac{\lambda_k}{\lambda_k - \lambda_j}, \quad n \in \mathbb{N},\ 1 \le j \le n; \qquad A_{1,1} = 1.$$

Here we assume that all parameters λ are different. The following equalities hold

$$A_{n+1,j} - A_{n,j} = \frac{\lambda_j}{\lambda_{n+1}} A_{n+1,j}, \quad n \in \mathbb{N},\ 1 \le j \le n. \tag{1.2}$$

Equivalently, functions Φ_n may be written as

$$\Phi_n(t) = L_n \sum_{j=1}^{n+1} \mathrm{e}^{-\lambda_j t}/\kappa_{n+1,j} \tag{1.3}$$

with the following notations: $L_n = \prod_{k=1}^{n} \lambda_k,\ n \in \mathbb{N}$, and $\kappa_{n+1,j} = \prod_{k=1,k\ne j}^{n+1}(\lambda_k - \lambda_j)$, $1 \le j \le n+1$, $n \in \mathbb{N}$; $L_0 = 1$, $\kappa_{1,1} = 1$. These notations will be also frequently used.

By using (1.2) one can easily verify the set of identities:

$$\frac{\mathrm{d}\Phi_n(t)}{\mathrm{d}t} \equiv -\lambda_{n+1}\Phi_n(t) + \lambda_n \Phi_{n-1}(t), \quad t \ge 0,\ n \in \mathbb{N}, \tag{1.4}$$

where $\Phi_0(t) = \mathrm{e}^{-\lambda_1 t},\ t \ge 0$, and $\Phi_n(0) = 0,\ n \in \mathbb{N}$.

2. Piecewise constant process

It is well known that for any $n \in \mathbb{N}$ the sum $T^{(n)} = T_1 + \cdots + T_n$ is Erlang-distributed. Precisely, if all λ's are different, $\lambda_k \ne \lambda_j$, for $k \ne j,\ k, j \in \mathbb{N}$, then the density function $\pi_n(t)$ of $T^{(n)}$ is

$$\pi_n(t) = \lambda_n \Phi_{n-1}(t) = \sum_{j=1}^{n} A_{n,j}\lambda_j \mathrm{e}^{-\lambda_j t}\mathbb{1}_{\{t\ge 0\}} = L_n \sum_{j=1}^{n} \frac{\mathrm{e}^{-\lambda_j t}}{\kappa_{n,j}} \mathbb{1}_{\{t\ge 0\}}, \quad n \in \mathbb{N}, \tag{2.1}$$

see Cox (1962). The usual modifications can be applied if two or more λ_j are equal.

Remark 2.1. Notice that $\pi_n(0) = 0$ and $\frac{\mathrm{d}^k \pi_n}{\mathrm{d}t^k}(0) = 0,\ n \ge 2,\ 1 \le k \le n-2$. This follows from the known Vandermonde properties, that is

$$\sum_{j=1}^{n} \lambda_j^{k+1} A_{n,j} = 0, \quad n \ge 2,\ 0 \le k \le n-2,$$

see e.g. Kuznetsov (2004, Corollary 1.1.1). Note, that $\sum_{j=1}^{n} A_{n,j} = 1,\ n \in \mathbb{N}$.