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Estimating the parameters of an α-stable distribution using the existence of moments of order statistics

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ABSTRACT

Necessary and sufficient conditions for the existence of order statistics moments of α-stable random variables are introduced. Using the obtained results, all parameters of α-stable distribution are estimated.

1. Introduction

The existence of moments for every distribution is an advantage for parameter estimation. There are several straightforward and efficient moment base estimators in classical statistics. However, lack of existence of the variance for non-Gaussian stable random variables is a drawback to introduce such an estimator. On the other hand, variance of many order statistics of an α-stable distribution exist. This is the main reason to make inference through the order statistics of a random sample of α-stable distribution. In other words, it can adapt well known order statistics-base estimators, such as L-estimator, best linear unbiased or invariant estimators for α-stable distribution parameters. In this paper, we prove the basic lemma about the existence of moments of stable distributions and propose new estimators using the results of the lemma.

A random variable X is said to have an α-stable (non-Gaussian α-stable) distribution if there are parameters $0 < \alpha < 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and a real number μ such that its characteristic function has the following form:

$$E\big(\exp(itX)\big) = \begin{cases} \exp\left\{-\sigma^\alpha \mid t\mid^\alpha \left(1 - i\beta(\text{sign } t)\tan\dfrac{\pi\alpha}{2}\right) + it\mu\right\} & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma \mid t \mid \left(1 + i\beta\dfrac{2}{\pi}(\text{sign } t)\log\mid t\mid\right) + it\mu\right\} & \text{if } \alpha = 1. \end{cases}$$

Since an α-stable distribution is characterized by four parameters, α, index of stability, σ, scale, β, skewness, and μ, location parameter, we will denote it by $S_\alpha(\sigma, \beta, \mu)$ and write $X \sim S_\alpha(\sigma, \beta, \mu)$ to indicate that the random variable X has α-stable distribution $S_\alpha(\sigma, \beta, \mu)$. From the characteristic function, $X \sim S_\alpha(\sigma, \beta, \mu)$ if and only if

$$X \overset{d}{=} \begin{cases} \sigma Z + \mu & \text{if } \alpha \neq 1, \\ \sigma Z + \mu + \beta\dfrac{2}{\pi}\sigma\log\sigma & \text{if } \alpha = 1, \end{cases} \tag{1}$$

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where $Z \sim S_\alpha(1, \beta, 0)$. The notation "$\stackrel{d}{=}$" denotes equality in distribution. An α-stable random variable is symmetric about μ if and only if $\beta = 0$, for more information on the basic concepts of α-stable random variables see Samorodnitsky and Taqqu (1994).

Class of α-stable distributions have been used for modeling in finance, telecommunications and medicine, see e.g. Kabašinskas et al. (2009), Nolan (2003) and Gallardo (2000). These applications provide motivations for estimating the parameters.

A few methods are proposed for estimating four parameters of α-stable distributions. McCulloch (1986) used sample quantiles for finding consistent estimators. Kogon and Williams (1998) estimated parameters by empirical characteristic function and Nolan (2001) computed maximum Likelihood estimators for $\alpha \geq 0.4$. Also, Antoniadis et al. (2006) proposed a wavelet based estimation method. In this article all parameters are estimated using the existence of moments of order statistics. We show that the presented estimators work well for small tail indexes through a simulation study. Also, it is shown that the presented estimator for $1/\alpha$ has a smaller asymptotic variance in comparison to some popular estimators.

In Section 2 necessary and sufficient conditions for the existence of order statistics moments of α-stable random variables are introduced. In Section 3, using the obtained results in Section 2, we try to estimate the α-stable distribution parameters and in Section 4 the obtained estimators are compared to some popular estimators. The paper is concluded in Section 5.

2. Moments of order statistics

For an α-stable random variable $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$, we have,

$$\lim_{\lambda \longrightarrow \infty} \lambda^\alpha P(X > \lambda) = C_\alpha \frac{1+\beta}{2} \sigma^\alpha,$$
$$\lim_{\lambda \longrightarrow \infty} \lambda^\alpha P(X < -\lambda) = C_\alpha \frac{1-\beta}{2} \sigma^\alpha, \tag{2}$$

where $C_\alpha = (1-\alpha)/(\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2}))$ if $\alpha \neq 1$ and $C_\alpha = \frac{2}{\pi}$ if $\alpha = 1$. This property is called regularly varying property, for proof see Samorodnitsky and Taqqu (1994). By the regularly varying property, we have the following theorem.

Theorem 2.1. *Let m be a positive number. Let $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$, $i = 1, \ldots, n$, be a sequence of independent α-stable random variables with $0 < \alpha < 2$ and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be its corresponding order statistics.*

(I) *Suppose $-1 < \beta_i < 1$, for $i = 1, \ldots, n$. In order that $EX_{k:n}^m$ exists it is necessary and sufficient that $\alpha^{-1}m < k < n + 1 - \alpha^{-1}m$.*

(II) *Suppose $\alpha \geq 1$ and $\beta_i = 1$ or $\beta_i = -1$, for $i = 1, \ldots, n$. In order that $EX_{k:n}^m$ exists it is sufficient that $\alpha^{-1}m < k < n + 1 - \alpha^{-1}m$.*

(III) *Suppose $\alpha < 1$ and $\beta_i = 1$ or $\beta_i = -1$, for $i = 1, \ldots, n$. In order that $EX_{k:n}^m$ exists it is necessary and sufficient that $k < n + 1 - \alpha^{-1}m$ or $\alpha^{-1}m < k$, respectively.*

Proof. It can be shown that

$$P(X_{k:n} > \lambda) = \sum_{j=n-k+1}^{n} (-1)^{j-(n-k+1)} \binom{j-1}{n-k} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} P(X_{i_1} > \lambda, \ldots, X_{i_j} > \lambda),$$
$$P(X_{k:n} < -\lambda) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} P(X_{i_1} < -\lambda, \ldots, X_{i_j} < -\lambda),$$

see Samorodnitsky (1986) for its proof. From independence and regularly varying property, equations (2), we have

$$\lim_{\lambda \longrightarrow \infty} \lambda^{(n-k+1)\alpha} P(X_{i_1} > \lambda, \ldots, X_{i_j} > \lambda) = \begin{cases} \prod_{h=1}^{n-k+1} \lim_{\lambda \to +\infty} \lambda^\alpha P(X_{i_h} > \lambda) & \text{if } j = n-k+1, \\ 0 & \text{if } j > n-k+1, \end{cases}$$
$$= \begin{cases} \prod_{h=1}^{n-k+1} C_\alpha \frac{1+\beta_{i_h}}{2} \sigma_{i_h}^\alpha & \text{if } j = n-k+1, \\ 0 & \text{if } j > n-k+1, \end{cases}$$

and

$$\lim_{\lambda \longrightarrow \infty} \lambda^{k\alpha} P(X_{i_1} < -\lambda, \ldots, X_{i_j} < -\lambda) = \begin{cases} \prod_{h=1}^{k} \lim_{\lambda \to +\infty} \lambda^\alpha P(X_{i_h} < -\lambda) & \text{if } j = k, \\ 0 & \text{if } j > k, \end{cases} = \begin{cases} \prod_{h=1}^{k} C_\alpha \frac{1-\beta_{i_h}}{2} \sigma_{i_h}^\alpha & \text{if } j = k, \\ 0 & \text{if } j > k. \end{cases}$$