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The randomly stopped geometric Brownian motion

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ABSTRACT

In this short note we compute the probability density function of the random variable X_T, where X_t is a geometric Brownian motion, and where T is a random variable independent of X_t and has either a Gamma distribution or it is uniformly distributed. In the last section of the note, the distribution obtained for X_T is fitted to the data consisting in the academic production of a set of mathematicians.

1. Introduction

As a model for the distribution of the empirical size of distinct quantities of economic interest, Reed (2001) has proposed the random variable X_T, where T is exponentially distributed with parameter λ, and X_t is the solution of the equation

$$X_t = X_0 + \mu' \int_0^t X_\xi \, d\xi + \sigma \int_0^t X_\xi \, dW_\xi . \tag{1}$$

In this equation, W_t is a standard Brownian motion, the first and second integrals on the right hand side are understood in the Riemann and Itô sense respectively, and μ' and σ are a drift and a diffusion coefficients. It is assumed that T is independent of the sigma algebra $\sigma_\tau\left(\left\{X_t : 0 \le t \le \tau\right\}\right)$ for all $\tau > 0$. The solution of Eq. (1) is the geometric Brownian motion and for a fixed t, the random variable X_t is log-normal distributed.

Reed found that X_T is distributed as a double-Pareto random variable with a density function given by

$$f(x) = \begin{cases} \dfrac{\gamma\beta}{x_0(\gamma+\beta)}\left(\dfrac{x}{x_0}\right)^{\beta-1} & \text{if } x < x_0, \\ \dfrac{\gamma\beta}{x_0(\gamma+\beta)}\left(\dfrac{x}{x_0}\right)^{-\gamma-1} & \text{if } x \ge x_0. \end{cases}$$

The double-Pareto random variable has three parameters γ, β and x_0 which depend on the drift and diffusion parameters μ' and σ and also on the parameter λ of the exponential distribution.

The purpose of this note is to extend the findings of Reed (2001) to the case where T is Gamma distributed with parameters α and λ. Thus, in Section 3 of this note, the distribution of X_T is determined when T has a Gamma distribution and where X_t is the value at time t of the solution of Eq. (1). In contrast to Reed's result, we found that the distribution of X_T is not determined by a power law, but that the density function of X_T has a tail with a power-like behavior, with an additional logarithmic term. This was to be expected, as Chen (2003) has shown that X_T has a power law only when T is exponentially distributed. It is to be noted that the case we study in Section 3 is a special case of the variance gamma process of

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mathematical finance, see Carr et al. (2002), Geman et al. (2001), Madan et al. (1998) and Applebaum (2009, p. 57). Rather than studying the distribution of $\log(X_T/X_0)$, as it is done in the above references, in Theorem 3 we obtain the distribution of X_T in terms of the Whittaker function, and leave X_0 as a free parameter to be estimated from the collected data. In Section 4 of this note, the distribution of X_T is determined when T has a uniform distribution. We found that in this case, the density function of X_T decreases faster than any power function, but slower than any exponential function.

In Section 5 we will fit the models obtained in Sections 3 and 4 to a data set consisting in the academic production of mathematicians who earned their degrees between 1940 and 1950 in an American University. For the determination of this population, we used the Mathematics Genealogy Project database. For each mathematician in this population, the length t of his academic life was determined by the difference between his last and first publications as recorded in the Mathematical Reviews Database of the American Mathematical Society. Also for each mathematician in the above time span, the number x_t of his publications was recorded as given by the same database. Here we will assume that x_t is a realization of X_T.

The writing of this note benefited significantly as a result of the remarks made by the referees. Particularly, we thank the referees for calling our attention to the fact that the Kolmogorov–Smirnov test does not hold when used with estimated parameters.

2. The Mellin transform

As we have already remarked above, the solution X_t of Eq. (1) has a log-normal distribution. More precisely,

$$X_t = X_0 \exp\left\{\left(\mu' - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \tag{2}$$

where W_t is the value at time t of a standard Brownian motion so that $\mathrm{var}(W_t) = t$ (Shreve, 2004, Example 4.4.8, p. 147). In the next theorem we follow Reed (2001) and determine the form of the Mellin transform of the density function of a random variable $\xi(T)$, where $\xi(t)$ is log-normal and T is a given random variable with h as its density function.

Theorem 1. *Let $\xi(t) = \exp\{\mu_0 + \mu t + \sigma\sqrt{t}\mathcal{Z}\}$ where $\mathcal{Z}$ is a standard normal random variable. Let T be a random variable with $h : [0, \infty) \to \mathbb{R}$ as its density function. Suppose that $\mathcal{Z}$ and T are independent. Let g be the density of $\xi(T)$ and $\hat{g}$ be the Mellin transform of g. Then we have*

$$\hat{g}(s) = e^{(s-1)\mu_0}\mathfrak{L}_h\left[(s-1)\left(\mu + (s-1)\frac{\sigma^2}{2}\right)\right],$$

where $\mathfrak{L}_h(x) = \int_0^\infty h(t)e^{xt}dt$ is the Laplace transform of h.

Proof. The probability density function of ξ_t is

$$f_t(x) = \frac{1}{\sqrt{2\pi t\sigma^2}x}\exp\left\{-\frac{\left(\log(x) - t\mu - \mu_0\right)^2}{2t\sigma^2}\right\}.$$

The Mellin transform of $f_t(x)$ is

$$\hat{f}_t(s) = \int_0^\infty f_t(x)x^{s-1}dx = \exp\left\{\frac{s-1}{2}\left(2t\mu + (s-1)t\sigma^2 + 2\mu_0\right)\right\}.$$

Since $\mathcal{Z}$ and T are independent, we have that the Mellin transform of g is

$$\begin{aligned}\hat{g}(s) &= \int_0^\infty h(t)\hat{f}_t(s)dt \\ &= e^{(s-1)\mu_0}\int_0^\infty h(t)\exp\left\{(s-1)\left(\mu + (s-1)\frac{\sigma^2}{2}\right)t\right\}dt \\ &= e^{(s-1)\mu_0}\mathfrak{L}_h\left[(s-1)\left(\mu + (s-1)\frac{\sigma^2}{2}\right)\right].\end{aligned}$$

This finishes the proof. □

Let us notice that the parameters X_0 and μ' of X_t in Eq. (2) and the parameters μ_0 and μ of $\xi(t)$ are related by

$$X_0 = e^{\mu_0} \quad \text{and} \quad \mu = \mu' - \frac{1}{2}\sigma^2, \tag{3}$$

while σ holds the same meaning in both X_t and $\xi(t)$.

3. Gamma life times

In this section we apply Theorem 1 to the case that T is a random variable with a Gamma distribution with parameters α and λ. The density function of T is $h(x) = \lambda(\lambda x)^{\alpha-1}e^{-\lambda x}/\Gamma(\alpha)$ and hence

$$\mathfrak{L}_h(s) = \left(1 - \frac{s}{\lambda}\right)^{-\alpha}$$