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The two-parameter Volterra multifractional process[☆]

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ABSTRACT

In the case where the parameters H_1 and H_2 belong to $(1/2, 1)$, Feyel and De La Pradelle (1991) have introduced a representation of the usual fractional Brownian sheet $\{B_{s,t}^{H_1,H_2}\}_{(s,t)\in\mathbb{R}_+^2}$, as a stochastic integral over the compact rectangle $[0, s] \times [0, t]$, with respect to the Brownian sheet. In this paper, we introduce the so-called two-parameter Volterra multifractional process by replacing in the latter representation of $\{B_{s,t}^{H_1,H_2}\}_{(s,t)\in\mathbb{R}_+^2}$ the constant parameters H_1 and H_2 by two Hölder functions $\alpha(s)$ and $\beta(t)$ with values in $(1/2, 1)$. We obtain that the pointwise and the local Hölder exponents of the two-parameter Volterra multifractional process at any point (s_0, t_0) are equal to $\min(\alpha(s_0), \beta(t_0))$.

1. Introduction

A fractional Brownian motion (fBm) $\{B_t^H, t \in \mathbb{R}_+\}$ with Hurst index $H \in (0, 1)$ is a real valued, centered Gaussian process with the covariance function given by

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \quad \forall s, t \in \mathbb{R}_+. \tag{1}$$

It was first introduced as a moving average Gaussian process, by Mandelbrot and Van Ness (1968). Fractional Brownian motion has interesting properties such as self-similarity of order $H \in (0, 1)$, stationary increments, and long range dependence which make it a good candidate in modeling different phenomena such as finance and telecommunication. However, this model may be restrictive due to the fact that all of its regularity and fractal properties are governed by the single Hurst parameter H. To model phenomena whose regularity evolves in time, e.g. Internet traffic or image processing, Lévy-Véhel and Peltier (1996) and Benassi et al. (1997) have independently introduced multifractional Brownian motion (mfBm) in terms of moving average representation and harmonizable representation, respectively. Multifractional Brownian motion is governed by a Hurst functional $H(t)$ with certain regularity in place of the constant Hurst parameter H in fBm. Several authors have studied sample path and statistical properties of mfBm. For example, Benassi et al. (1997) considered the sample path Hölder regularity of mfBm and determined the Hausdorff dimension of its graph. Ayache et al. (2002a) and Herbin (2006) studied the covariance structure of mfBm from its harmonizable representation. There are multiparameter extensions of fBm, among which two typical ones are multiparameter Lévy fBm and fractional Brownian sheets, where

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the former is isotropic while the latter ones are anisotropic in general. Since they were introduced by Kamont (1996) (see also Ayache et al., 2002b), fractional Brownian sheets (fBs) have been studied extensively as a representative of anisotropic Gaussian random fields in recent years. See, for example, Dunker (2000), Mason and Shi (2001), Øksendal and Zhang (2000), Xiao and Zhang (2002), Ayache and Xiao (2005), Ayache et al. (2008) and Wu and Xiao (2007) and the references therein for further information. Still, the regularity of fBs does not evolve in the 2-dimensional "time" parameter $t \in \mathbb{R}_+^2$. To model anisotropic Gaussian random fields whose regularity evolves in time, such as images, Ayache and Léger (1999) and Herbin (2006) introduced so-called multifractional Brownian sheets (mfBs) in terms of their moving average representations and harmonizable representations, where the constant Hurst vector of fBs is substituted by Hurst functionals. Furthermore, they showed that mfBs have a continuous modification and determined the pointwise and local Hölder exponents of mfBs. They also proved that mfBs are locally self-similar. We refer to Ayache and Léger (1999) and Herbin (2006) for the definitions of the corresponding concepts and results. In the case where the parameters H_1 and H_2 belong to $(1/2, 1)$, Feyel and De La Pradelle (1991) have introduced a representation of fractional Brownian sheet $\{B_{s,t}^{H_1,H_2}\}_{(s,t)\in\mathbb{R}_+^2}$ as a stochastic integral over the compact rectangle $[0, s]\times[0, t]$ with respect to the Brownian sheet. To model anisotropic Gaussian random fields whose regularity evolves in time, such as images, we introduce in this paper the so-called, two-parameter Volterra multifractional process, by replacing, in the latter representation of $\{B_{s,t}^{H_1,H_2}\}_{(s,t)\in\mathbb{R}_+^2}$, the constant parameters H_1 and H_2, by two Hölder functions $\alpha(s)$ and $\beta(t)$ with values in $(1/2, 1)$. It seems possible to construct a stochastic integral with respect to this process by using Malliavin calculus. This process is different from processes defined by Ayache and Léger (1999) and Herbin (2006) because they do not have the same covariance functions. The remainder of this article is organized as follows. In Section 2, we define the two-parameter Volterra multifractional process and show that it has continuous modification. We also study its covariance function. In Section 3, we establish the locally self-similar property for the two-parameter Volterra multifractional process. In Section 4, we show that the pointwise and the local Hölder exponents of the two-parameter Volterra multifractional process at any point (s_0, t_0) are equal to $\min(\alpha(s_0), \beta(t_0))$.

2. Definition of the two-parameter Volterra multifractional process

It is well-known (see Feyel and De La Pradelle, 1991) that the fractional Brownian sheet B^{H_1,H_2} with Hurst parameter $(H_1, H_2) \in (1/2, 1)^2$ can be represented for $(s, t) \in \mathbb{R}_+^2$ as

$$B_{s,t}^{H_1,H_2} = \int_0^s \int_0^t K_{H_1}(s, u)K_{H_2}(t, v)W(du, dv)$$

where $K_{H_1}(s, u) = c_{H_1} u^{1/2-H_1} \int_u^s y^{H_1-1/2}(y-u)^{H_1-3/2}dy$

with $c_{H_1} = \left(\dfrac{\pi H_1(2H_1-1)}{\Gamma(2-2H_1)\Gamma(H_1+1/2)^2 \sin(\pi(H_1-1/2))}\right)^{1/2} (H_1 - 1/2)$,

$K_{H_2}(t, v) = c_{H_2} v^{1/2-H_2} \int_v^t y^{H_2-1/2}(y-v)^{H_2-3/2}dy$

with $c_{H_2} = \left(\dfrac{\pi H_2(2H_2-1)}{\Gamma(2-2H_2)\Gamma(H_2+1/2)^2 \sin(\pi(H_2-1/2))}\right)^{1/2} (H_2 - 1/2)$,

and $W(du, dv)$ is a Brownian sheet.

Let a and b be two real numbers satisfying $1/2 < a < b < 1$. Throughout the paper, we consider two functions $\alpha : \mathbb{R}_+ \to [a, b]$ and $\beta : \mathbb{R}_+ \to [a, b]$. We assume that α is a η-Hölder function and β is a θ-Hölder function with $\sup_{s\in\mathbb{R}_+} \alpha(s) < \eta < 1$ and $\sup_{t\in\mathbb{R}_+} \beta(t) < \theta < 1$ where η and θ are real numbers.

Definition 2.1. The two-parameter Volterra multifractional process is the centered Gaussian process $B^{\alpha(.),\beta(.)} = \{B_{s,t}^{\alpha(s),\beta(t)}, (s, t) \in \mathbb{R}_+\}$ given by the following Volterra-type representation

$$B_{s,t}^{\alpha(s),\beta(t)} = \int_0^s \int_0^t K_{\alpha(s)}(s, u)K_{\beta(t)}(t, v)W(du, dv) \tag{2}$$

where $K_{\alpha(s)}(s, u) = u^{1/2-\alpha(s)} \int_u^s y^{\alpha(s)-1/2}(y-u)^{\alpha(s)-3/2}dy$

and $K_{\beta(t)}(t, v) = v^{1/2-\beta(t)} \int_v^t y^{\beta(t)-1/2}(y-v)^{\beta(t)-3/2}dy$.

If $\alpha(.) = H_1$ and $\beta(.) = H_2$ are two constants, $B_{.}^{H_1,H_2}$ is a fractional Brownian sheet up to the multiplicative constant $c_{H_1}c_{H_2}$. Before establishing properties of $B_{.}^{\alpha(.),\beta(.)}$, we state a lemma regarding an estimate on K_λ that we use throughout the paper.