Statistical Methodology 6 (2009) 513–526

Contents lists available at ScienceDirect

Statistical Methodology

journal homepage: www.elsevier.com/locate/stamet

Non-asymptotic sequential confidence regions with fixed sizes for the multivariate nonlinear parameters of regression

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ARTICLE INFO

Article history:
Received 25 October 2008
Received in revised form
10 May 2009
Accepted 12 May 2009

Keywords:
Non-asymptotic nonlinear estimation
Sequential analysis
Confidence regions
Guaranteed accuracy

ABSTRACT

In this paper we consider a sequential design for the estimation of nonlinear parameters of regression with guaranteed accuracy. Non-asymptotic confidence regions with fixed sizes for the least squares estimates are used. The obtained confidence region is valid for finite numbers of data points when the distributions of the observations are unknown.

1. Introduction

In this paper we investigate some non-asymptotic properties of the least squares estimates for nonlinear parameters of regression. Lately there has been considerable interest in nonlinear models in different practical fields, especially in economics. The primary reason behind this interest is the fact that linear models, while having a wide variety of practical applications, did not quite meet the expectations. The asymptotic properties of nonlinear least squares estimates are well investigated and discussed [1,5–8,12,14]. At the same time, only few results addressing the finite sample properties exist, whereas the non-asymptotic solution for the problem of the parameter estimation for regression is practically important because the sample volume is always limited from above. Non-asymptotic estimation of the scalar parameter of nonlinear regression by means of confidence regions was examined by Timofeev [12]. A similar estimation of the multivariate parameter was researched by Timofeev [13]. In this paper a sequential design is suggested that will make it possible to solve the problem of nonlinear estimation of multivariate parameter of regression by means of confidence

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1572-3127/$ – see front matter
doi:10.1016/j.stamet.2009.05.002

regions in the non-asymptotic setting. As opposed to the method suggested by Timofeev [13], the solution presented in this paper does not use explicit expressions for the gradients of the loss function. The solution was obtained under the condition that distributions of observations are unknown. The mean observation time was estimated in the suggested sequential design.

2. Statement of the problem

A stochastic process $\mathbf{X} = \{X(k) | k \geq 0\}$ satisfies the following equation

$$X(k) = A(k, \theta^*) + \xi(k), \quad k \geq 1, \tag{1}$$

where $X(k), A(k, \theta^*), \xi(k) \in R^m$. The nonlinear functions $\{A(k, \theta^*) \,|k \geq 0\}$ are defined. A vector sequence $\{\xi(k) \,|k \geq 1\}$ with unknown distribution is such that

$$\forall k \geq 0 : \left(\mathbf{E}\,(\xi(k)) = 0, \mathbf{E}\left(\xi(k)\xi(k)^{\mathrm{T}}\right) = L(k)\right),$$

where $\{L(k) \,|k \geq 0\}$ is a known sequence of non-random diagonal matrixes $m \times m$ that can be described as follows:

$$\forall k \geq 0 : L(k) = \operatorname{diag}\left(l_1^2(k), \ldots, l_m^2(k)\right), \quad \exists K : \forall_{j=1}^m l_j^2(k) < K < \infty.$$

An unknown parameter θ^* belongs to a closed ball Θ embedded in an m-dimensional Euclidean space R^m.

Let $\Delta = \{\delta_1, \delta_2, \ldots, \delta_m\} \subseteq R^1, \forall \delta_j \in \Delta : \delta_j \in]0, \infty[$ be a set of required dimensions of the confidence parallelepiped. $P_c \in]0, 1[$ is the required value of the confidence coefficient.

We need to develop a sequential design for confidence estimation of the parameter θ^* that would determine:

- the stochastic stopping time $\tau \geq 1$,
- a rule for building a confidence rectangular parallelepiped $\Xi(\tau)$ in the compact Θ,

that meet the following conditions:

$$\mathbf{P}_{\theta^*}\left(\theta^* \in \Xi(\tau)\right) \geq P_c, \qquad \mathbf{P}_{\theta^*}(\tau < \infty) = 1,$$
$$\forall \theta_1, \theta_2 \in \Xi(\tau) : \forall_{j=1}^m \left|\langle\theta_1\rangle_j - \langle\theta_2\rangle_j\right| \leq \delta_j, \quad \delta_j \in \Delta.$$

From now on, $\|\cdot\|$ stands for a norm of the space in which the compact Θ is embedded. $\langle a\rangle_i$ from now on stands for the ith component of the vector a. The brackets will be omitted when no ambiguity arises. For the sake of clarity, the following shorthand notation will be used throughout the rest of the paper: $\mathbf{P}$ (instead of $\mathbf{P}_\theta$) and $\mathbf{E}$ (instead of $\mathbf{E}_\theta$).

3. Solution method

Let us assume that stochastic vector functions $\{A(k, \theta) \,|k \geq 0\}$ on Θ meet the following condition:

$$\forall\,(\theta_1, \theta_2 \in \Theta, n \geq 1) : \sum_{k=0}^{n-1} R_k\,(\theta_1, \theta_2)\,(nm)^{-1} > 0. \tag{2}$$

Here $R_k\,(\theta_1, \theta_2) = [A(k, \theta_1) - A(k, \theta_2)]\, L^{-1}(k)\, [A(k, \theta_1) - A(k, \theta_2)]^{\mathrm{T}}, R_k\,(\theta_1, \theta_2) \in R^1$. Let $\Theta^* = \{\theta_n | n \geq 1\}$ be a sequence of estimators of the parameter θ^* which is defined as follows:

$$\forall n \geq 1 : \theta_n = \arg \inf_{\theta \in \Theta} I(n, \theta) \tag{3}$$

where $I(n, \theta) = \sum_{k=1}^{t(n)} [X(k) - A(k, \theta)]\, L^{-1}(k)\, [X(k) - A(k, \theta)]^{\mathrm{T}}\,(t(n)m)^{-1}$.

Here $\{t(n)\} \subseteq N, N = \{0, 1, 2, \ldots\}, \lim_{n\to\infty} t(n) \to \infty, I(n, \theta) \in R^1$. For each $n \geq 1$ define a functional $\Phi\,(n, \theta, \theta_n)$ and a closed set sequence $\{\Xi\,(n) \,|n \geq 1\} \subseteq \Theta$ so that

$$\Phi\,(n, \theta, \theta_n) = I(n, \theta) - I(n, \theta_n), \quad \forall\,(n \geq 1, \theta \in \Xi\,(n)) : \Phi\,(n, \theta, \theta_n) \leq c(n).$$