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A generalised Student's t-distribution

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ABSTRACT

We introduce a natural extension of the Student's t-distribution that also allows for a negative shape parameter or more commonly referred to as the degrees of freedom of this distribution. This distribution unifies all types of tail decay and allows extra flexibility in the kurtosis of the t-distribution. We illustrate the use of this distribution with an application to pharmaceutical data.

1. Introduction

The Student's t-distribution is a heavy-tailed distribution that arises naturally in the construction of hypothesis tests for the expected value of a normally distributed random sample of observations with unknown variance (see Kotz et al. (1995)). For location and scale standardised, its probability density function f is proportional to

$$f(x) \propto \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \tag{1}$$

where $x \in \mathbb{R}$ and $\nu > 0$. The parameter ν controls the decay of the tail, i.e., large values of ν yield distributions with near exponential tails and small values of ν produce heavy-tailed distributions. For example, as ν tends to infinity, the standard normal distribution is obtained whereas for $\nu \le k$, the k-th moment of this distribution does not exist. Owing to the fact that the t-distribution contains the normal distribution as a special case, its use has been of great importance in the general data modelling framework as an alternative and more robust probability model to the normal distribution. As a consequence there exist a number of extensions of the t-distribution, some of which for example, introduce an additional parameter to the distribution and therefore result in more flexible models, e.g., McDonald and Newey (1988), Hansen (1994) and Azzalini and Capitanio (2003). Other distributions with similar properties to the aforementioned distributions exist in the literature such as the distributions introduced by Johnson et al. (1980) and Walker (1999). However, these distributions do not admit a simple closed-form expression for their density function and this hampers their applicability for classical inference.

As we will show, the Student's t-distribution can be naturally extended to cover also the case $\nu < 0$, leading thus to a symmetric probability distribution that unifies all types of tail decay and allows extra flexibility in the kurtosis of the distribution. To our knowledge, this unified distribution does not exist in the literature and yet it allows one to model short-tailed symmetric data under the t-distributional framework as well as to account for the uncertainty in the nature of the tail of the distribution.

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The main difference between our proposed model and other distributions that appear in the literature is that our model incorporates a broad range of distributions defined on bounded supports whereas the aforementioned distributions are defined on $(-\infty, \infty)$ or include bounded distributions as limiting cases only. For example, the McDonald and Newey (1988) class of distributions encompasses heavy-tailed distributions and incorporates as a limiting case the Box and Tiao (1973) power-exponential distribution which has exponential tails. Therefore, for these types of tail decay the aforementioned models are natural competitors of the t-distribution. In comparison, the proposed model of this paper can capture heavy tailed, exponential and short-tailed symmetric distributions.

The paper is structured as follows. In Section 2, the generalised t-distribution is presented. The moments and excess kurtosis of the distribution are derived and the multivariate analogue of this probability is distribution given. In Section 3 we illustrate a biostatistical dataset where the observed variable is well modelled by a short-tailed t-distribution. We also compare our proposed model with McDonald and Newey (1988) model and Box and Tiao (1973) model which is a special case of the former. Finally, in Section 4 we give some concluding remarks.

2. The generalised t-distribution

2.1. Unified probability model

To establish the generalisation of the t-distribution for the case $\nu < 0$, we specify initially the support, say $\mathcal{S}_\nu$, of this distribution. From expression (1), the density is non negative when

$$x \in \mathcal{S}_\nu := \begin{cases} \{x : -\sqrt{-\nu} \le x \le \sqrt{-\nu}\} & \nu < 0, \\ \{x : -\infty < x < \infty\} & \nu > 0. \end{cases}$$

Integrating expression (1) over the support $\mathcal{S}_\nu$ we obtain the normalising constant of this distribution, i.e.,

$$\int_{\mathcal{S}_\nu} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx = \begin{cases} \dfrac{\sqrt{-\nu\pi}\,\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right)}{\Gamma\left(1 - \frac{\nu}{2}\right)} & \nu < 0, \\ \dfrac{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} & \nu > 0, \end{cases}$$

which can be unified in a single expression, i.e.,

$$\int_{\mathcal{S}_\nu} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx = \sqrt{|\nu|}\mathrm{Be}\left\{\frac{1}{2}, \frac{2\nu - 1 + \mathrm{sign}(\nu)}{4\,\mathrm{sign}(\nu)}\right\}, \quad \nu \in \mathbb{R} \setminus \{0\}. \tag{2}$$

On dividing the right hand side of expression (1) by the normalising constant in Eq. (2), we obtain the probability density function of the t-distribution that allows for negative ν, a case that corresponds to finite lower and upper end points for this distribution.

To avoid the discontinuity at $\nu = 0$ we reparametrise this model under $\xi = 1/\nu$. This reparameterisation is natural as the parameter ξ shows the decay of the tail of the distribution, i.e., $\xi > 0$ implies heavy-tails, $\xi \to 0$ corresponds to the standard normal distribution which has exponential tails and $\xi < 0$ implies short-tails with finite end points. By incorporating location and scale parameters $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ to this probability distribution, we have the following definition.

Definition 1. A random variable X is said to follow the generalised t-distribution with location, scale and shape parameters $(\mu, \sigma, \xi) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, short-hand $X \sim gt(\mu, \sigma, \xi)$, if the distribution function $F_X(x)$ and the density function $f_X(x)$ of X, $x \in \mathbb{R}$, are given by

$$F_X(x) = \frac{1}{2} + \frac{\sqrt{|\xi|}(x - \mu)}{\sigma \mathrm{Be}\left\{\frac{1}{2}, \frac{\xi - (\xi - 2)\mathrm{sign}(\xi)}{4\xi}\right\}}\,{}_2F_1\left[\frac{1}{2}, \frac{1+\xi}{2\xi}; \frac{3}{2}; \min\left\{-\xi\left(\frac{x-\mu}{\sigma}\right)^2, 1\right\}\right], \tag{3}$$

$$f_X(x) = \frac{\sqrt{|\xi|}}{\sigma \mathrm{Be}\left\{\frac{1}{2}, \frac{\xi - (\xi - 2)\mathrm{sign}(\xi)}{4\xi}\right\}} \left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)^2\right\}_+^{-\frac{1+\xi}{2\xi}}, \tag{4}$$

respectively, where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function (Abramowitz and Stegun, 1965) and $x_+ = \max(x, 0)$.

The generalised t-distribution of Definition 1 introduces a natural extension of the t-distribution. When $\xi < 0$ the support of this distribution, say $\mathcal{S}_{\mu,\sigma,\xi}$, is given by $\mathcal{S}_{\mu,\sigma,\xi} = (\mu - \sigma/\sqrt{|\xi|}, \mu + \sigma/\sqrt{|\xi|})$, whereas when $\xi > 0$ the distribution coincides with the location-scale t-distribution with $\nu = 1/\xi$ degrees of freedom. For $\xi = 0$, taken as $\xi \to 0$, the distribution function of X is the normal distribution with mean μ and variance σ^2. Table 1 and Fig. 1 illustrate the effect of the shape parameter ξ on the distribution and density function respectively, with $\mathcal{I}(x \in \mathcal{S}_{\mu,\sigma,\xi})$ denoting the indicator function for real x.