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Quantile based entropy function in past lifetime

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ABSTRACT

Di Crescenzo and Longobardi (2002) introduced a measure of uncertainty in past lifetime distributions and studied its relationship with residual entropy function. In the present paper, we introduce a quantile version of the entropy function in past lifetime and study its properties. Unlike the measure of uncertainty given in Di Crescenzo and Longobardi (2002) the proposed measure uniquely determines the underlying probability distribution. The measure is used to study two nonparametric classes of distributions. We prove characterizations theorems for some well known quantile lifetime distributions.

1. Introduction

Let X be an absolutely continuous random variable (rv) representing the lifetime of a component with cumulative distribution function (CDF) $F(t) = P(X \leq t)$ and survival function (SF) $\bar{F}(t) = P(X > t) = 1 - F(t)$. The measure of uncertainty (Shannon, 1948) is defined by

$$\xi(X) = \xi(f) = -\int_0^\infty (\log f(x)) f(x) dx = -E(\log f(X)), \tag{1}$$

where $f(t)$ is the probability density function (PDF) of X. Eq. (1) gives the expected uncertainty contained in $f(t)$ about the predictability of an outcome of X, which is known as the Shannon information measure. The length of time during a study period has been considered as a prime variable of interest in many fields such as reliability, survival analysis, economics, business, etc. In such cases, the information measures are functions of time and thus they are dynamic. Based on this idea, Ebrahimi (1996) defined the residual Shannon entropy of X at time t by

$$\xi(X; t) = \xi(f; t) = -\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right) \log\left(\frac{f(x)}{\bar{F}(t)}\right) dx. \tag{2}$$

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Note that $\xi(X; t) = \xi(X_t)$, where $X_t = (X - t \mid X > t)$ is the residual time associated to X (see also Ebrahimi and Pellerey, 1995).

However, in many realistic situations, uncertainty is related to the past lifetime rather than the residual lifetime. For example, if we consider a system which had already failed at time t, then its uncertainty is related to the past, *i.e.*, on which instant $(0, t)$ it has failed. Motivated by this, Di Crescenzo and Longobardi (2002) introduced the past entropy as

$$\begin{aligned}\overline{\xi}(X; t) = \overline{\xi}(f; t) &= -\int_0^t \left(\frac{f(x)}{F(t)}\right) \log\left(\frac{f(x)}{F(t)}\right) dx,\\ &= \log F(t) - \frac{1}{F(t)} \int_0^t (\log f(x)) f(x) dx,\end{aligned} \tag{3}$$

where $\overline{\xi}(X; t) = \overline{\xi}(X^t)$ and $X^t = (t - X \mid X \leq t)$ is the past lifetime associated to X. Denoting $a(x) = f(x)/F(x)$ the reversed hazard rate (see Block et al., 1998), Eq. (3) can be rewritten as

$$\overline{\xi}(X; t) = 1 - \frac{1}{F(t)} \int_0^t \log(a(x)) f(x) dx. \tag{4}$$

Given that at time t, a unit is found to be down, $\overline{\xi}(X; t)$ measures the uncertainty about its past lifetime. Interesting extensions and multivariate forms of measures of uncertainty are also available in literature. For recent works on entropy in past lifetime we refer to Di Crescenzo and Longobardi (2002, 2006), Nanda and Paul (2006) and Kundu et al. (2010).

All these theoretical results and applications thereof are based on the distribution function. A probability distribution can be specified either in terms of the distribution function or by the quantile functions (QF). Recently, it has been shown by many authors that quantile functions

$$Q(u) = F^{-1}(u) = \inf\{t \mid F(t) \geq u\}, \quad 0 \leq u \leq 1 \tag{5}$$

are efficient and equivalent alternatives to the distribution function in modeling and analysis of statistical data (see Gilchrist, 2000; Nair and Sankaran, 2009). In many cases, QF's are more convenient as they are less influenced by extreme observations and thus provide a straightforward analysis with a limited amount of information. For detailed and recent studies on QF, its properties and usefulness in the identification of models we refer to Nair et al. (2008, 2011), Nair and Sankaran (2009), Sankaran and Nair (2009), Sankaran et al. (2010) and the references therein.

Many of the quantile functions used in applied work like various forms of lambda distributions (Ramberg and Schmeiser, 1974; Freimer et al., 1988; van Staden and Loots, 2009; Gilchrist, 2000), the power-Pareto distribution (Gilchrist, 2000, Hankin and Lee, 2006), Govindarajulu distribution (Nair et al., 2011) etc. do not have tractable distributions. This makes the analytical study of the properties of $\xi(X)$ of these distributions by means of (1) difficult. Thus a formulation of the definition and properties of entropy function in terms of quantile functions is called for. Such a discussion has several advantages. Analytical properties of the entropy function obtained in this approach can be used as alternative tools in modeling data. Sometimes, the quantile-based approach is better in terms of tractability. New models and characterizations that are unresolvable in the distribution function approach can be resolved with the aid of QF's. In view of these, the objective of the present work is to initiate a discussion of entropy function in terms of QF's. The present paper introduces the Shannon entropy function for past lifetime using the QF's and proved some useful characterization theorems arising out of it. Unlike the entropy function in past lifetime due to Di Crescenzo and Longobardi (2002), an explicit expression is obtained for quantile based entropy function in past lifetime.

The rest of the paper is organized as follows. In Section 2, we consider some useful reliability measures in terms of quantile function and introduce the quantile based Shannon entropy in past lifetime. Various properties of the measure are discussed. Finally, Section 3 proves some characterization results based on the measures considered in Section 2.

2. Quantile based entropy in past lifetime

When F is continuous, we have from (5), $FQ(u) = u$, where $FQ(u)$ represents the composite function $F(Q(u))$. Defining the density quantile function by $fQ(u) = f(Q(u))$ (see Parzen, 1979) and quantile density function by $q(u) = Q'(u)$, where the prime denotes the differentiation, we have

$$q(u) fQ(u) = 1. \tag{6}$$

The reversed hazard rate function $a(x)$ can be expressed in terms of the quantile function, is given by

$$A(u) = aQ(u) = a(Q(u)) = u^{-1} fQ(u) = [uq(u)]^{-1}.$$

The function $A(u)$ is referred to as reversed hazard quantile function. Following Nair and Sankaran (2009), $A(u)$ explains the conditional probability of failure in the past small interval of time given that the failure occurred prior to the $100u\%$ point of distribution. Like $a(x)$ that determines the CDF or SF uniquely, $A(u)$ also uniquely determines the QF by the identity

$$Q(u) = \int_0^u \frac{dt}{tA(t)}.$$