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Stochastic order characterization of uniform integrability and tightness

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ABSTRACT

We show that a family of random variables is uniformly integrable if and only if it is stochastically bounded in the increasing convex order by an integrable random variable. This result is complemented by proving analogous statements for the strong stochastic order and for power-integrable dominating random variables. In particular, we show that, whenever a family of random variables is stochastically bounded by a p-integrable random variable for some $p > 1$, there is no distinction between the strong order and the increasing convex order. These results also yield new characterizations of relative compactness in Wasserstein and Prohorov metrics.

1. Introduction

Let $\{X_n\}$ be a sequence of random variables such that $X_n \to X$ almost surely. Lebesgue's classical dominated convergence theorem implies that

$$E|X_n - X| \to 0 \tag{1}$$

if there exists an integrable random variable Y such that

$$|X_n| \le Y \quad \text{almost surely for all } n.$$

A probabilistic version (e.g. Thorisson (2000, Theorem 9.1)) of the above condition is to require that

$$|X_n| \le_{\text{st}} Y \quad \text{for all } n, \tag{2}$$

where $\le_{\text{st}}$ denotes the strong (also known as usual) stochastic order (Müller and Stoyan, 2002; Shaked and Shanthikumar, 2007; Leskelä, 2010). It is well known that in general (2) is not necessary for (1). We will show that a sharp characterization can be obtained when the strong stochastic order in (2) is replaced by a weaker one. Namely, we will show that (1) holds if and only if there exists an integrable random variable Y such that

$$|X_n| \le_{\text{icx}} Y \quad \text{for all } n, \tag{3}$$

where $\le_{\text{icx}}$ denotes the increasing convex stochastic order (Müller and Stoyan, 2002; Shaked and Shanthikumar, 2007).

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$\{\lvert X_\alpha\rvert^q\}$ is st-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is st-bounded by a q-integrable r.v.	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is icx-bounded by a q-integrable r.v.
		$\Downarrow$		
$\{X_\alpha\}$ is uniformly q-integrable	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert^q\}$ is icx-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\mu_\alpha\}$ is rel. compact in the q-Wasserstein metric
		$\Downarrow$		
		$\{X_\alpha\}$ is bounded in L^q	$\Leftrightarrow$	$\{\mu_\alpha\}$ is bounded in the q-Wasserstein metric
		$\Downarrow$		
		$\{\lvert X_\alpha\rvert\}$ is st-bounded by an integrable r.v.		
		$\Downarrow$		
$\{X_\alpha\}$ is uniformly integrable	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is icx-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\mu_\alpha\}$ is rel. compact in the 1-Wasserstein metric
		$\Downarrow$		
		$\{X_\alpha\}$ is bounded in L^1	$\Leftrightarrow$	$\{\mu_\alpha\}$ is bounded in the 1-Wasserstein metric
		$\Downarrow$		
$\{\lvert X_\alpha\rvert^p\}$ is st-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is st-bounded by a p-integrable r.v.		
		$\Downarrow$		
$\{X_\alpha\}$ is uniformly p-integrable	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert^p\}$ is icx-bounded by an integrable r.v.		
		$\Downarrow$		
		$\{X_\alpha\}$ is bounded in L^p		
		$\Downarrow$		
$\{X_\alpha\}$ is tight	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is st-bounded by a finite r.v.	$\Leftrightarrow$	$\{\mu_\alpha\}$ is rel. compact in the Prohorov metric

Fig. 1. Stochastic boundedness relationships.

More generally, our first main result, Theorem 1, shows that a family of random variables $\{X_n\}$ is uniformly integrable if and only if (3) holds. For almost surely convergent random sequences, this result yields the equivalence of (1) and (3) (e.g. Kallenberg (2002, Proposition 4.12)). From the analysis point of view, the characterization of uniform integrability in terms of the increasing convex order can be seen as a new way to represent domination in Lebesgue's dominated convergence theorem in the natural manner. What makes this result important in probability is that convex and increasing convex orders are intimately connected with the existence of martingales and submartingales with given marginals (Kellerer, 1972; Hirsch and Yor, 2010); see also Remark 6.

The second main result, Theorem 2, shows that, when studying whether a family of random variables is stochastically bounded by a random variable in L^p for some $p > 1$, there is no need to distinguish between the strong order and the increasing convex order. This somewhat surprising result, which is a consequence of a Hardy–Littlewood maximal inequality, may open new ways to establishing strong stochastic bounds using tools of convex analysis.

The main results are complemented by Proposition 1, which states that $\{X_n\}$ is tight if and only if (2) holds for some (almost surely finite) random variable Y. This simple result is probably well known, because 'bounded in probability' and 'stochastically bounded' are commonly used as synonyms for 'tight'. We have formulated it here explicitly in order to complete the big picture on various stochastic boundedness relationships in Theorem 3. The implication diagram in Fig. 1 summarizes the findings and illustrates how these concepts are related to relative compactness with respect to Wasserstein and Prohorov metrics. Fig. 1 provides a new unified view of earlier studies on lattice properties of increasing convex and strong stochastic orders (Kertz and Rösler, 2000; Müller and Scarsini, 2006).

The rest of this paper is organized as follows. Section 2 introduces definitions and notation. Section 3 discusses uniform integrability and tightness of positive random variables, and Section 4 extends the analysis to power-integrable random variables. Section 5 summarizes the main results in a diagram (Fig. 1) and presents counterexamples confirming the sharpness of the implications in the diagram.