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Notes on entropic convergence and the weak entropy inequality

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ABSTRACT

Based on the criteria of the algebraic convergence for entropy inequality, the algebraic convergence result for the entropy of a Ginzburg–Landau model is presented. The relationship between the weak entropy inequality and the weak Poincaré inequality, tensorization and perturbation of the weak entropy inequality are discussed.

1. Introduction

The functional inequalities imply plentiful analytic properties of Markov semigroups and generators, which are related to various behaviors of corresponding Markov processes. For instance, the entropy inequality is equivalent to the exponential convergence in relative entropy for Markov process. The rate of convergence to equilibrium is one of the main problems in the theory of Markov process. Algebraic convergence of transition function to equilibrium in the L^2 sense were studied by Liggett (1991), Deuschel (1987), Chen and Wang (2003). Furthermore, Röckner and Wang (2001) introduced a weak Poincaré inequality(WPI) to describe general convergence rates of the semigroups.

In aspects of the algebraic convergence in the entropy sense, Bertini and Zegarlinski (1999) gave a logarithmic Nash inequality for Kawasaki dynamics which implies algebraic decay of the entropy. Cattiaux et al. (2007) studied a weakened form of logarithmic Sobolev inequality which is used for deriving entropic convergence. Gao and Li (2008) gave a criteria of algebraic convergence for entropy.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $(\mathcal{L}, D(\mathcal{L}))$ a densely defined linear operator which generates a symmetric Markov semigroup P_t on $L^2(\mu)$. Define the entropy

$$Ent_\mu(f) = \begin{cases} \mu(f\log f) - \mu(f)\log\mu(f) & f \geq 0 \text{ and } f\log f \in L_1(\mu), \\ +\infty & \text{otherwise.} \end{cases}$$

The Dirichlet form is determined by

$$\mathcal{E}(f, g) = \mu(-f(\mathcal{L}g)), \quad \forall f, g \in D(\mathcal{L}).$$

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Theorem 1.1 (Gao and Li, 2008). *Take $1 < p, q < \infty$ be two real numbers such that $p^{-1} + q^{-1} = 1$ and let V be a nonnegative function from $L_1(\mu)$ to $[0, +\infty]$. Consider the following two statements:*

(a) *There exists a constant $C > 0$ such that for all $0 \le f \in D(\mathcal{L})$ satisfying $\mu(f) = 1$ and $\log f \in D(\mathcal{L})$,*

$$Ent_\mu(f) \le C\mathcal{E}(f, \log f)^{1/p}[V(f)]^{1/q}. \tag{1.1}$$

(b) *There exists a constant $C > 0$ such that for all $0 \le f \in D(\mathcal{L})$ with $\mu(f) = 1$ and $\log f \in D(\mathcal{L})$,*

$$Ent_\mu(P_t f) \le CV(f)/t^{q-1}. \tag{1.2}$$

Then the following conclusions hold:

(i) *Assume that $V(P_t f) \le V(f)$ for all $f \in L_1(\mu)$ and $t > 0$. Then (1.1) holds if and only if*

$$Ent_\mu(P_t f) \le \frac{q^{q-1}C^q V(f)}{\left(pt + \left(\frac{V(f)}{Ent_\mu(f)}\right)^{1/(q-1)} qC^p\right)^{q-1}}, \tag{1.3}$$

for all $0 \le f \in D(\mathcal{L})$ with $\mu(f) = 1$ and $\log f \in \mathcal{D}(\mathcal{L})$, where $\frac{0}{0} := 0$. In particular, (a) *implies* (b).

(ii) *Assume that there exists a constant $B > 0$ such that $\mathcal{E}(P_t f, \log P_t f) \le B\mathcal{E}(f, \log f)$ for all $f \in D(\mathcal{L})$ with $\log f \in D(\mathcal{L})$ and all $t > 0$. Then* (b) *implies* (a).

Additionally, Gao and Li (2008) improved an algebraic convergence result for the entropy of an exclusion process and introduced the following weak entropy inequality (WEI) as an extension of the traditional entropy inequality:

$$\mu(f \log f) \le \beta(r)\mathcal{E}(f, \log f) + rM(f), \quad r > 0,\ \mu(f) = 1,\ f \in D(\mathcal{L})_+,\ \log f \in D(\mathcal{L}), \tag{1.4}$$

where β is a nonnegative and decreasing function on $(0, \infty)$ and M is a nonnegative function from $L_1(\mu)$ to $[0, +\infty]$ and $D(\mathcal{L})_+ = \{f \in D(\mathcal{L}); f \ge 0\}$.

In this paper, we will apply the Theorem 1.1 to the Ginzburg–Landau model. Then the relationship between the weak entropy inequality and weak Poincáre inequality is presented. At last, we introduce the tensorization and perturbation of the weak entropy inequality.

2. The entropy estimator of the Ginzburg–Landau model

In recent years, the rate of convergence to equilibrium has attracted the attention of many authors in the context of particle systems. Janvresse et al. (1999) estimated the decay to equilibrium in the variance sense of zero-range dynamics. Landim and Yau (2003) proved the convergence to equilibrium of Ginzburg–Landau process in L_2 sense.

For each $x \in \mathbb{Z}^d$, let η_x represent the continuous charge at site x. The dynamics of the charge configuration are given by the infinitesimal generator (Landim and Yau, 2003)

$$\mathcal{L} = \frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}}(\partial_{\eta_x} - \partial_{\eta_y})^2 - \frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}}(H'(\eta_y) - H'(\eta_x))(\partial_{\eta_y} - \partial_{\eta_x}),$$

where $H : \mathbb{R} \to \mathbb{R}$ is some potential function and ∂_{η_x} means the partially differentiate.

The associated Dirichlet form is given by

$$\mathcal{E}(f, f) = \sum_{x,y\in\mathbb{Z}^d,|x-y|=1} E_{\nu_\rho}\left[\left(\frac{\partial f}{\partial \eta_x} - \frac{\partial f}{\partial \eta_y}\right)^2\right],$$

where the invariant measures are product measure $\{\nu_\rho, \rho \in \mathbb{R}\}$ with marginal $E_{\nu_\rho}(\eta_0) = \rho$ and the process is reversible with respect to this family of measures and where E_{ν_ρ} stands for expectation with respect to ν_ρ.

For $1 < p < \infty$ and $1 < q < \infty$, define the triple norm

$$|||f|||_{p,q} = \left(\sum_{x\in\mathbb{Z}^d}\left\|\frac{\partial f}{\partial \eta_x}\right\|_p^q\right)^{1/q},$$

where $\|f\|_p$ is the standard L_p norm.

Theorem 2.1. *For any $\rho \in \mathbb{R}$, there exists some constant C such that for all local smooth functions $f \in D(\mathcal{L})_+$ satisfying $\mu(f) = 1$, $\log f \in D(\mathcal{L})$ and $f \log f \in L_1(\nu_\rho)$,*

$$Ent_{\nu_\rho}[P_t f] \le CV(f)t^{-\frac{d}{2}},$$

where $V(f) = \sup_{s\ge 0} |||\sqrt{P_s f}|||_{2,1}^2$.