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Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space

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ABSTRACT

In this note we prove an existence and uniqueness result of mild solutions for a neutral stochastic differential equation with finite delay, driven by a fractional Brownian motion in a Hilbert space and we establish some conditions ensuring the exponential decay to zero in mean square for the mild solution.

1. Introduction

In this paper, we study the existence, uniqueness and asymptotic behavior of mild solutions for the following neutral stochastic differential equation with finite delay:

$$\begin{aligned} &d[x(t)+g(t,x(t-r(t)))]=[Ax(t)+f(t,x(t-\rho(t)))]dt+\sigma(t)dB^H(t), \quad 0\le t\le T,\\ &x(t)=\varphi(t), \quad -\tau\le t\le 0 \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X, B^H is a fractional Brownian motion on a real and separable Hilbert space Y, $r, \rho : [0, +\infty) \to [0, \tau]$ $(\tau > 0)$ are continuous and $f, g : [0, +\infty) \times X \to X$, $\sigma : [0, +\infty] \to \mathcal{L}_2^0(Y, X)$ are appropriate functions. Here $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert–Schmidt operators from Y into X (see Section 2).

We would like to mention that the theory for the stochastic differential equations (without delay) driven by a fractional Brownian motion (fBm) have recently been studied intensively (see e.g. Coutin and Qian (2000), Tindel et al. (2003), Nualart and Răşcanu (2002), Nualart and Ouknine (2002), Hu and Nualart (2007) and Nualart and Saussereau (2009) and the references therein).

As, for the stochastic functional differential equations driven by an fBm, even much less has been done, as far as we know, there exist only a few papers published in this field. In Ferrante and Rovira (2006), the authors studied the existence and regularity of the density by using the Skorohod integral based on the Malliavin calculus. Neuenkirch et al. (2008) studied

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the problem by using rough path analysis. Ferrante and Rovira (2010) studied the existence and convergence when the delay goes to zero by using the Riemann–Stieltjes integral. Using also the Riemann–Stieltjes integral, Boufoussi and Hajji (2011) and Boufoussi et al. (2011) proved the existence and uniqueness of a mild solution and studied the dependence of the solution on the initial condition in finite and infinite dimensional space. Very recently, Caraballo et al. (2011) have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions by using Wiener integral.

On the other hand, to the best of our knowledge, there is no paper which investigates the study of neutral stochastic differential equations with delays driven by a fractional Brownian motion. Thus, we will make the first attempt to study such problem in this paper. Our results are inspired by the one in Caraballo et al. (2011) where the existence and uniqueness of mild solutions to model (1) with $g = 0$ is studied, as well as some results on the weak solution and asymptotic behavior.

The rest of this paper is organized as follows, In Section 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we recall some preliminary results about analytic semi-groups and fractional power associated to its generator. The existence and uniqueness of mild solutions are discussed in Section 3 by using Banach fixed point theorem. In Section 4, we investigate the exponential asymptotic behavior for the mild solution obtained in Section 3.

2. Preliminaries

In this section we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian and we recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators which will be used throughout the whole of this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s)d\beta(s) \tag{2}$$

where $\beta = \{\beta(t) : t \in [0, T]\}$ is a Wiener process, and $K_H(t; s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K_H(t, s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi\rangle_{\mathcal{H}} = H(2H - 1) \int_0^T \int_0^T \psi(s)\varphi(t)|t - s|^{2H-2} dsdt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^*\varphi)(s) = \int_s^T \varphi(r)\frac{\partial K}{\partial r}(r, s)dr.$$

We refer to Nualart (2006) for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0, T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^*\varphi)(t)d\beta(t).$$

It follows from Nualart (2006) that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)\|\psi(t)\|s - t|^{2H-2} dsdt < \infty$$