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On the convergence of LePage series in Skorokhod space

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ABSTRACT

We consider the problem of the convergence of the so-called LePage series in the Skorokhod space $\mathbb{D}^d = \mathbb{D}([0,1],\mathbb{R}^d)$ and provide a simple criterion based on the moments of the increments of the random process involved in the series. This provides a simple sufficient condition for the existence of an α-stable distribution on $\mathbb{D}^d$ with given spectral measure.

1. Introduction

We are interested in the convergence in the Skorokhod space $\mathbb{D}^d = \mathbb{D}([0,1],\mathbb{R}^d)$ endowed with the J_1-topology of random series of the form

$$X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0,1], \tag{1}$$

where $\alpha \in (0,2)$ and:

- $(\Gamma_i)_{i\geq 1}$ is the increasing enumeration of the points of a Poisson point process on $[0,+\infty)$ with Lebesgue intensity;
- $(\varepsilon_i)_{i\geq 1}$ is an i.i.d. sequence of real random variables;
- $(Y_i)_{i\geq 1}$ is an i.i.d. sequence of $\mathbb{D}^d$-valued random variables;
- the sequences (Γ_i), (ε_i) and (Y_i) are independent.

Note that a more constructive definition for the sequence $(\Gamma_i)_{i\geq 1}$ is given by

$$\Gamma_i = \sum_{j=1}^{i} \gamma_j, \quad i \geq 1,$$

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where $(\gamma_i)_{i\geq 1}$ is an i.i.d. sequence of random variables with exponential distribution of parameter 1, and independent of (ε_i) and (Y_i).

Series of the form (1) are known as LePage series. For fixed $t \in [0, 1]$, the convergence in $\mathbb{R}^d$ of the series (1) is ensured as soon as one of the following conditions is satisfied:

- $0 < \alpha < 1, \mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$,
- $1 \leq \alpha < 2, \mathbb{E}\varepsilon_1 = 0, \mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$.

Here $|.|$ denotes the usual Euclidean norm on $\mathbb{R}$ or on $\mathbb{R}^d$. The random variable $X(t)$ has then an α-stable distribution on $\mathbb{R}^d$. Conversely, it is well known that any α-stable distributions on $\mathbb{R}^d$ admits a representation in terms of LePage series (see for example Samorodnitsky and Taqqu (1994), Section 3.9).

There is a vast literature on symmetric α-stable distributions on separable Banach spaces (see e.g. Ledoux and Talagrand (1991), Araujo and Giné (1980)). In particular, any symmetric α-stable distribution on a separable Banach space can be represented as an almost surely convergent LePage series (see Corollary 5.5 in Ledoux and Talagrand (1991)). The existence of a symmetric α-stable distribution with a given spectral measure is not automatic and is linked with the notion of stable type of a Banach space; see Theorem 9.27 in Ledoux and Talagrand (1991) for a precise statement. Davydov et al. (2008) consider α-stable distributions in the more general framework of abstract convex cones.

The space $\mathbb{D}^d$ equipped with the norm

$$\|x\| = \sup\{|x_i(t)|,\ t \in [0, 1],\ i = 1, \dots, d\}, \quad x = (x_1, \dots, x_d) \in \mathbb{D}^d,$$

is a Banach space but is not separable. The uniform topology associated with this norm is finer than the J_1-topology. On the other hand, the space $\mathbb{D}^d$ with the J_1-topology is Polish, i.e. there exists a metric on $\mathbb{D}^d$ compatible with the J_1-topology that makes $\mathbb{D}^d$ a complete and separable metric space. However, such a metric cannot be compatible with the vector space structure since the addition is not continuous in the J_1-topology. These properties explain why the general theory of stable distributions on separable Banach space cannot be applied to the space $\mathbb{D}^d$.

Nevertheless, in the case where the series (1) converges, the distribution of the sum X defines an α-stable distribution on $\mathbb{D}^d$. We can determine the associated spectral measure σ on the unit sphere $\mathbb{S}^d = \{x \in \mathbb{D}^d;\ \|x\| = 1\}$. It is given by

$$\sigma(A) = \frac{\mathbb{E}\Big(|\varepsilon_1|^\alpha \|Y_1\|^\alpha \mathbf{1}_{\{\mathrm{sign}(\varepsilon_1)Y_1/\|Y_1\|\in A\}}\Big)}{\mathbb{E}(|\varepsilon_1|^\alpha \|Y_1\|^\alpha)}, \quad A \in \mathcal{B}(\mathbb{S}^d).$$

This is closely related to regular variations theory (see Hult and Lindskog (2006), Davis and Mikosch (2008)). For all $r > 0$ and $A \in \mathcal{B}(\mathbb{S}^d)$ such that $\sigma(\partial A) = 0$, it holds that

$$\lim_{n\to\infty} n\mathbb{P}\left(\frac{X}{\|X\|} \in A \mid \|X\| > rb_n\right) = r^{-\alpha}\sigma(A),$$

with

$$b_n = \inf\{r > 0;\ \mathbb{P}(\|X\| < r) \leq n^{-1}\}, \quad n \geq 1.$$

The random variable X is said to be regularly varying in $\mathbb{D}^d$ with index α and spectral measure σ.

In this framework, convergence of the LePage series (1) in $\mathbb{D}^d$ is known in some particular cases only:

- When $0 < \alpha < 1, \mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}\|Y_1\|^\alpha < \infty$, the series (1) converges almost surely uniformly in $[0, 1]$ (see example 4.2 in Davis and Mikosch (2008)).
- When $1 \leq \alpha < 2$, the distribution of the ε_i's is symmetric, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $Y_i(t) = \mathbf{1}_{[0,t]}(U)$ with $(U_i)_{i\geq 1}$ an i.i.d. sequence of random variables with uniform distribution on $[0, 1]$, the series (1) converges almost surely uniformly on $[0, 1]$ and the limit process X is a symmetric α-stable Lévy process (see Rosiński (2001)).

The purpose of this note is to complete these results and to provide a general criterion for almost sure convergence in $\mathbb{D}^d$ of the random series (1). Our main result is the following:

Theorem 1. *Suppose that* $1 \leq \alpha < 2$,

$$\mathbb{E}\varepsilon_1 = 0, \qquad \mathbb{E}|\varepsilon_1|^\alpha < \infty \quad \text{and} \quad \mathbb{E}\|Y_1\|^\alpha < \infty.$$

Suppose furthermore that there exist $\beta_1, \beta_2 > \frac{1}{2}$ *and* F_1, F_2 *nondecreasing continuous functions on* $[0, 1]$ *such that, for all* $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}|Y_1(t_2) - Y_1(t_1)|^2 \leq |F_1(t_2) - F_1(t_1)|^{\beta_1}, \tag{2}$$

$$\mathbb{E}|Y_1(t_2) - Y_1(t)|^2|Y_1(t) - Y_1(t_1)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}. \tag{3}$$

Then, the LePage series (1) *converges almost surely in* $\mathbb{D}^d$.

The proof of this theorem is detailed in the next section. We provide hereafter a few cases where Theorem 1 can be applied.