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Quantile based entropy function

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ABSTRACT

Quantile functions are efficient and equivalent alternatives to distribution functions in modeling and analysis of statistical data (see Gilchrist, 2000; Nair and Sankaran, 2009). Motivated by this, in the present paper, we introduce a quantile based Shannon entropy function. We also introduce residual entropy function in the quantile setup and study its properties. Unlike the residual entropy function due to Ebrahimi (1996), the residual quantile entropy function determines the quantile density function uniquely through a simple relationship. The measure is used to define two nonparametric classes of distributions.

1. Introduction

In recent years, there has been a great interest in the measurement of uncertainty of probability distributions. Let X be a nonnegative absolutely continuous random variable (rv) representing the lifetime of a component with cumulative distribution function (CDF) $F(t) = P(X \leq t)$ and survival function (SF) $\bar{F}(t) = P(X > t) = 1 - F(t)$. The measure of uncertainty defined by Shannon (1948) was

$$\xi(X) = \xi(f) = -\int_0^\infty (\log f(x)) f(x) dx = -E(\log f(X)), \tag{1}$$

where $f(t)$ is the probability density function (PDF) of X. Eq. (1) gives the expected uncertainty contained in $f(t)$ about the predictability of an outcome of X, which is known as Shannon entropy measure. Length of time during a study period has been considered as a prime variable of interest in many fields such as reliability, survival analysis, economics, business, etc. In such cases, the information measures are functions of time, thus they are dynamic. Based on this idea, Ebrahimi (1996) defined the residual Shannon entropy of X at time t as

$$\begin{aligned} \xi(X;t) = \xi(f;t) &= -\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right) \log\left(\frac{f(x)}{\bar{F}(t)}\right) dx, \\ &= \log \bar{F}(t) - \frac{1}{\bar{F}(t)} \int_t^\infty (\log f(x)) f(x) dx. \end{aligned} \tag{2}$$

Note that $\xi(X;t) = \xi(X_t)$, where $X_t = (X - t \mid X > t)$ is the residual time associated to X. By writing $h(t) = f(t)/\bar{F}(t)$, the failure rate function of X, (2) can equivalently be written as

$$\xi(X;t) = 1 - \frac{1}{\bar{F}(t)} \int_t^\infty (\log h(x)) f(x) dx. \tag{3}$$

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A similar function can be obtained in terms of the inactivity time $(t - X \mid X \le t)$ given in Di Crescenzo and Longobardi (2002). Interesting extensions and multivariate forms are also available in the literature. For additional information on these measures, we refer to Belzunce et al. (2004), Ebrahimi (1996), Ebrahimi and Kirmani (1996), Ebrahimi and Pellerey (1995), Nanda and Paul (2006) and Sunoj et al. (2009).

All these theoretical results and applications thereof are based on the distribution function. A probability distribution can be specified either in terms of the distribution function or by the quantile functions (QFs). Recently, it has been showed by many authors that QFs

$$Q(u) = F^{-1}(u) = \inf\{t \mid F(t) \ge u\}, \quad 0 \le u \le 1, \tag{4}$$

are efficient and equivalent alternatives to the distribution function in modeling and analysis of statistical data (see Gilchrist, 2000; Nair and Sankaran, 2009). In many cases, QFs are more convenient as they are less influenced by extreme observations and thus provide a straightforward analysis with a limited amount of information. For a detailed and recent study on QF, its properties and its usefulness in the identification of models we refer to Lai and Xie (2006), Nair and Sankaran (2009), Nair et al. (2011), Sankaran and Nair (2009), Sankaran et al. (2010) and the references therein.

Although variety of research is available for various measures of uncertainty, a study of the same using QF does not appear to have been taken up. Also, many QFs used in applied works such as various forms of lambda distributions (Ramberg and Schmeiser, 1974; Freimer et al., 1998; Gilchrist, 2000; van Staden and Loots, 2009), the power-Pareto distribution (Gilchrist, 2000; Hankin and Lee, 2006), Govindarajulu distribution (Nair et al., 2011) etc. do not have tractable distribution functions. This makes the statistical study of the properties of $\xi(X)$ for these distributions by means of (1) difficult. Thus a formulation of the definition and properties of entropy function in terms of QFs is called for. Such a discussion has several advantages. Analytical properties of the entropy function obtained in this approach can be used as an alternative tool in modeling statistical data. Sometimes the quantile based approach is better in terms of tractability. New models and characterizations that are unresolvable in the distribution function approach can be resolved with the aid of quantile approach. Further, an explicit relationship between quantile entropy function and quantile density function in residual time can be derived.

The paper is organized as follows. In Section 2, we discuss some useful reliability measures in terms of quantile function. We introduce Shannon entropy function and residual entropy function in quantile setup and study their properties. Section 3 presents characterization results for certain lifetime quantile models based on the residual quantile entropy function.

2. Quantile based Shannon entropy

When F is continuous, we have from (4), $FQ(u) = u$, where $FQ(u)$ represents the composite function $F(Q(u))$. Defining the density quantile function by $fQ(u) = f(Q(u))$ (see Parzen, 1979) and quantile density function by $q(u) = Q'(u)$, where the prime denotes the differentiation, we have

$$q(u)fQ(u) = 1. \tag{5}$$

The hazard rate quantile function is defined by,

$$H(u) = hQ(u) = h(Q(u)) = (1-u)^{-1}fQ(u) = [(1-u)q(u)]^{-1}. \tag{6}$$

Following Nair and Sankaran (2009), $H(u)$ explains the conditional probability of failure in the next small interval of time given survival until $100(1-u)\%$ point of distribution. Like $h(t)$ that determines the CDF or SF uniquely, $H(u)$ also uniquely determines the QF by

$$Q(u) = \int_0^u \frac{dt}{(1-t)H(t)}.$$

From (5), the Shannon entropy defined in (1) can be written in terms of QF as

$$\xi(X) = \xi = \int_0^1 (\log q(p))\, dp. \tag{7}$$

Clearly, by knowing either $Q(u)$ or $q(u)$, the expression for $\xi(X)$ is quite simple to compute. An equivalent definition for the residual entropy (2) in terms of QF is given by

$$\xi Q(u) = \xi(X; Q(u)) = \log(1-u) + (1-u)^{-1} \int_u^1 (\log q(p))\, dp. \tag{8}$$

From (3), we can also write (8) as

$$\xi Q(u) = 1 - (1-u)^{-1} \int_u^1 \log H(p) dp. \tag{9}$$

$\xi Q(u)$ measures the expected uncertainty contained in the conditional density about the predictability of an outcome of X until $100(1-u)\%$ point of distribution. Further, differentiating Eq. (8) with respect to u, we get

$$\xi' Q(u) = -\frac{1}{(1-u)} + \frac{1}{(1-u)^2} \int_u^1 \log q(p) dp - \frac{1}{(1-u)} \log q(u),$$