Statistics and Probability Letters 82 (2012) 1120–1128

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Bounds on exponential moments of hitting times for reflected processes on the positive orthant

Chihoon Lee *

Department of Statistics, Colorado State University, Fort Collins, CO 80523-1877, USA

ARTICLE INFO

Article history:
Received 5 December 2011
Accepted 23 February 2012
Available online 5 March 2012

MSC:
primary 60G22
secondary 90B18
60G15
60G18

Keywords:
Reflected fractional Brownian motion
Reflected Lévy process
Heavy traffic theory
First hitting times
Exponential moments

ABSTRACT

We first consider a multi-dimensional reflected fractional Brownian motion process on the positive orthant with the Hurst parameter $H \in (0, 1)$. In particular, when $H > 1/2$, this model serves to approximate fluid stochastic network models fed by a big number of heavy tailed ON/OFF sources in heavy traffic. Assuming the initial state lies outside some compact set, we establish that the exponential moment of the first hitting time to the compact set has a lower bound with an exponential growth rate in terms of the magnitude of the initial state. We extend this result to the case for reflected processes driven by a class of stable Lévy motions, which arise as approximations to cumulative network traffic over a time period. For the case of $H = 1/2$, under a natural stability condition on the reflection directions and drift vector, we obtain a matching upper bound on exponential moments of hitting times, which grows at an exponential rate in terms of the initial condition of the process. We also show that such an upper bound is valid for reflected processes driven by general light-tailed Lévy processes.

1. Introduction

We first consider a multi-dimensional reflected fractional Brownian motion process $\{Z(t) : t \geq 0\}$ (abbreviated as RFBM hereafter) on the positive orthant $S := \mathbb{R}^d_+$, with drift $r^0 \in \mathbb{R}^d$ and the Hurst parameter $H \in (0, 1)$. For example, when the Hurst parameter H is greater than $1/2$, the RFBM model serves as an approximation for stochastic networks with a large number of heavy-tailed ON/OFF sources (Delgado, 2007), or with long range dependent arrival and service time processes (Majewski, 2005) in the heavy traffic regime (with the dimension d being the number of nodes or servers). This model was further studied in the subsequent papers (Delgado, 2008, 2010). Besides Delgado (2007), the multi-dimensional RFBM process has also been obtained as an approximating model of the queue-length process in Konstantopoulos and Lin (1996), in which a single-class queueing network with long-range dependent arrival and service processes is considered. Recently, stationarity and control problems of a tandem fluid network with fractional Brownian motion input (a two-dimensional RFBM model in the first quadrant) were studied in Lee and Weerasinghe (2011) via a coupling time approach. From the viewpoint of the applications, it is of great importance to study the basic time behavior of the RFBM process. However, relatively little has been known about the properties of RFBM, since a fractional Brownian motion (FBM) is neither a semi-martingale nor a Markov process, many techniques from the classical theory of stochastic calculus are inapplicable to its analysis.

* Tel.: +1 970 491 7321; fax: +1 970 491 1084.
E-mail address: chihoon@stat.colostate.edu.
URL: http://www.stat.colostate.edu/~chihoon.

0167-7152/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2012.02.022

In this work, we first establish a lower bound growth estimate on the expected exponential first hitting time for RFBM process $\{Z(t) : t \geq 0\}$ on the orthant S; more precisely, it is proven that for any $\beta \in (0, \infty)$ there exists a $\gamma \in (0, \infty)$ such that

$$\mathbb{E}_x[e^{\beta \tau_B}] \geq e^{\gamma |x|} \quad \text{for all } x \in S \setminus D, \tag{1}$$

where $\tau_B := \inf\{t \geq 0 : Z(t) \in B\}$ for some compact sets B and D in S, centered at the origin, satisfying $B \subset D$ and a constant $\gamma = \gamma(H) > 0$, which depends on the Hurst parameter $H \in (0, 1)$. Here (and throughout the paper), $\mathbb{E}_x$ denotes the expectation conditional on the process Z starting from $x \in S$. The result (1) implies that the expectation of the hitting time random variable τ_B has *at least* a linear growth rate, proportional to the distance of the initial state from the origin. We refer the reader to Lee (2011b,a) for the related uniform return time (to some compact sets) results and a basic geometric drift inequality result of the RFBM process on the positive orthant. Our analysis reveals that such a lower bound result can be extended to the case for reflected processes driven by a class of stable Lévy motions, which arise as approximations to cumulative network traffic over a time period (cf. Mikosch et al. (2002) and the references therein).

Next, when $H = 1/2$, we obtain a matching upper bound on exponential moments of hitting times, which grows also at an exponential rate in terms of the initial condition of the process. Namely, we prove under a natural stability condition on the reflection directions and drift vector that there exist $\beta \in (0, \infty)$ and $\alpha_1, \alpha_2 \in (0, \infty)$ such that

$$\mathbb{E}_x[e^{\beta \tau_B}] \leq \alpha_1 e^{\alpha_2 |x|} \quad \text{for all } x \in S \setminus D, \tag{2}$$

where the sets B and D satisfy the same conditions as before. We were unable to establish an upper bound estimate (2) for a general $H \in (0, 1)$, mainly due to the lack of (strong) Markov and independent incremental properties of FBM when $H \neq 1/2$ (see Remark 3.8). However, we establish that such an upper bound is valid for reflected processes driven by general light-tailed Lévy processes; see Corollary 3.9.

The study of the first hitting times of reflected processes is important both in its own right and for the sake of its applications to several areas such as engineering and finance; we refer the reader to Chapter 3 in Harrison (1985), Chapter 7 in Karlin and Taylor (1975), and Budhiraja and Lee (2007) for several examples of related applications. However, the computation of the moments of hitting times functionals driven by FBMs, seems to be quite challenging in general. For example, commonly used numerical techniques using suitable Markov chain approximations (cf. Chapter 7, Section 4 of Ethier and Kurtz (1986)) for the reflected processes (and then using the transition matrix of the chain to approximate the probability distribution, and hence the moments of the hitting time) are inapplicable, since FBM is not Markov. Also, a linear programming method introduced in Helmes et al. (2001), which is based on the martingale characterization of wide range of stochastic processes via its generator, does not apply to here either, since FBM is not a martingale.

The organization of the paper is as follows. In Section 2, we carefully describe our model in Definition 2.1 and make a standard assumption on reflection matrix (see **(HR0)** in Section 2), which is used in heavy traffic analysis for invoking a functional central limit theorem in Delgado (2007, 2008). We present the main results in Section 3. The proof of the lower bound estimate (Theorem 3.1) is based on oscillation inequality for reflected processes together with a maximal inequality for FBM with $H \in (0, 1)$, which leads to an appropriate tail probability estimate of the running supremum of the RFBM process (see (10)) in terms of its initial condition. An explicit computation on the lower bound for the case of simple one-dimensional example is then provided. The upper bound estimate (Theorem 3.7) is obtained from a tail probability estimate with exponential decay rates (24), which is proved by making use of a suitable Lyapunov function $T(\cdot) : S \to [0, \infty)$ and its basic properties. Also, an exponential moment estimate (22) on the running supremum of the standard Brownian motion increments is crucially used in the proof.

We use the following notation. The set of positive integers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and non-negative real numbers by $\mathbb{R}_+$. For $a \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and for $x \in \mathbb{R}^d$ the max-norm of x, i.e., $\max_{1 \leq i \leq d} |x_i|$, will be denoted by $|x|$. Let $\mathbb{I} = \mathbb{I}_{d \times d}$ denote the identity matrix for some $d \in \mathbb{N}$. When it is clear from the context, we will omit the subscript. For a set $A \subseteq \mathbb{R}^d$, denote its interior and boundary by A° and ∂A, respectively, and $1_C(\cdot)$ will denote an indicator function on a set C. For sets $A, B \subseteq \mathbb{R}^d$, $\text{dist}(A, B)$ will denote the distance between the two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. Let $\mathcal{D}(X, Y)$ denote the class of right continuous functions with having left limit defined from X to Y, equipped with the usual Skorokhod topology. Inequalities for vectors are interpreted componentwise.

2. Model and assumptions

We begin by the definitions of multi-dimensional FBM and RFBM. Let $d \in \mathbb{N}$. A stochastic process $B_H = \{B_H(t) = (B_H^{(1)}(t), \ldots, B_H^{(d)}(t))^T, t \geq 0\}$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, is called a d-dimensional FBM of Hurst parameter $H \in (0, 1)$, starting from $B_H(0) \in \mathbb{R}^d$, and associated matrix Λ, if it is a continuous Gaussian process with initial condition $B_H(0)\mathbb{P}$-a.s. and with covariance function given by

$$\text{Cov}(B_H(t), B_H(s)) = \mathbb{E}((B_H(t) - B_H(0))(B_H(s) - B_H(0))^T) = \Lambda_H(s, t)\Lambda,$$

for any $s, t \geq 0$. Here, Λ is a $d \times d$ positive definite matrix and

$$\Lambda_H(s, t) := \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$