Statistics and Probability Letters 82 (2012) 1160–1168

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A note on the domination inequalities and their applications

Yaofeng Ren *, Jing Shen

Department of Applied Mathematics, College of Science, The Naval University of Engineering, 430033 Wuhan, PR China

ARTICLE INFO

Article history:
Received 16 January 2012
Received in revised form 4 March 2012
Accepted 4 March 2012
Available online 10 March 2012

MSC:
60E15
60J60
60G44
60G40

Keywords:
Domination inequality
Increasing process
Locally square integrable martingale
Ornstein–Uhlenbeck process
Bessel process

ABSTRACT

In this note, we present some refinements of the well-known domination inequalities. Let X be an adapted positive cadlag process dominated by a predictable increasing process A with $A_0 = 0$. We derive some sharper constants in the inequalities. For the widely used inequality

$$E\left[(X_\infty^*)^p\right] \leq \frac{2-p}{1-p} E\left(A_\infty^p\right), \quad 0 < p < 1,$$

we obtain the following strengthened version

$$E\left[(X_\infty^*)^p\right] \leq \frac{1}{1-p}\left(\frac{1}{p}\right)^p E\left(A_\infty^p\right), \quad 0 < p < 1.$$

Where $X_\infty^* = \sup_{t<\infty} |X_t|$. This inequality is sharper in the sense that it yields that the growth rate of C_p is $1/p$ as $p \to 0^+$ in the following L_p inequality:

$$\|X_\infty^*\|_p \leq C_p \|A_\infty\|_p, \quad 0 < p < 1.$$

We also apply the improved inequalities to martingales, the Ornstein–Uhlenbeck process and Bessel processes.

1. Introduction

Since established by Lenglart (1977) and improved by Yor (1979, 1982), domination inequalities have become an important tool in probability and statistics. Domination inequalities are often used to establish inequalities between a pair of stochastic processes, such as inequalities for semi-martingales (Barlow and Yor, 1982), inequalities for local times of martingales (Chao and Chou, 2000), inequalities for Ornstein–Uhlenbeck processes (Graversen and Peskir, 2000), inequalities for diffusion processes (Botnikov) and some others (Veraar and Weis, 2010; Banuelos and Baudoin, 2011). Though well known and widely used for a long time, domination inequalities can be further improved. In this paper, we present some refinements of domination inequalities. We shall use the standard notions of general theory of stochastic processes, and thus consider stochastic processes with cadlag paths. We first introduce some notations and recall some well-known results. For more details, refer to Revuz and Yor (1998) or Liptser and Shiryaev (1989).

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying usual conditions. A stochastic process $A = (A_t)_{t\geq 0}$ is called an increasing process if it is adapted to the family $(\mathcal{F}_t)$, whose paths are positive, increasing, finite and right continuous on $[0, +\infty)$. An adapted positive cadlag process X is called dominated by an adapted increasing process A with $A_0 \geq 0$, if

$$E(X_\tau) \leq E(A_\tau)$$

for any bounded stopping time τ.

* Corresponding author.
E-mail address: renyaofeng@sina.com (Y. Ren).

0167-7152/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2012.03.002

For real numbers a and b, we denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. Let X be a random variable. For $p \in (0, \infty)$, we denote $\|X\|_p = [E(|X|^p)]^{\frac{1}{p}}$ the L_p norm of X. For a stochastic process $X = (X_t)_{t\geq 0}$ and a stopping time τ, we write

$$X_\tau^* = \sup_{s\leq\tau} |X_s|, \qquad X_\infty^* = \sup_{s<\infty} |X_s|.$$

Lenglart (1977) established the following domination inequalities or domination principles.

Lemma 1 (*Lenglart*). *Let X be an adapted positive cadlag process dominated by a predictable increasing process A with $A_0 = 0$. Then for any stopping time τ, any constants $c > 0, d > 0$,*

$$P(X_\tau^* \geq c) \leq \frac{1}{c} E(A_\tau \wedge d) + P(A_\tau \geq d). \tag{1}$$

Lemma 2 (*Lenglart*). *Let X be an adapted positive cadlag process dominated by a predictable increasing process A with $A_0 = 0$. Then for $0 < p < 1$, the following inequality holds for any stopping time τ*

$$E[(X_\tau^*)^p] \leq \frac{2-p}{1-p} E(A_\tau^p). \tag{2}$$

From Lenglart's proof, Yor (1982) gave the following inequality.

Lemma 3 (*Yor, 1982*). *Let X be an adapted positive cadlag process dominated by a predictable increasing process A with $A_0 = 0$. Then for any constants $c > 0, d > 0$,*

$$P(X_\infty^* \geq c, A_\infty \leq d) \leq \frac{1}{c} E(A_\infty \wedge d). \tag{3}$$

Let μ be a positive Radon measure on R_+^2, define $F_\mu(z) = \mu([0, x] \times [0, y]),\ z = (x, y) \in R_+^2$. For a continuous increasing function $\phi(x)$ from R_+ to R_+ with $\phi(0) = 0$, set

$$\Phi(x) = \phi(x) + x \int_x^\infty \frac{d\phi(u)}{u}.$$

If $\phi(x) = x^p (0 < p < 1)$, then

$$\Phi(x) = \frac{1}{1-p} x^p.$$

Yor (1982) established a general form of the domination inequality and deduced some interesting results from it.

Lemma 4 (*Yor*). *Let X be an adapted positive cadlag process and be dominated by a predictable increasing process A with $A_0 = 0$.*

(1) *Let μ be a positive Radon measure on R_+^2. Then*

$$E\left[F_\mu\left(X_\infty^*, \frac{1}{A_\infty}\right)\right] \leq E\left[2F_\mu\left(A_\infty, \frac{1}{A_\infty}\right) + A_\infty \int_{\left(x>A_\infty, y\leq\frac{1}{A_\infty}\right)} x^{-1}\mu(dz) + \int_{\left(y>\frac{1}{A_\infty}\right)} (xy \vee 1)^{-1} \mu(dz)\right]. \tag{4}$$

(2) *If $\phi(x)$ and $\psi(x)$ are right-continuous increasing functions from R_+ to R_+, then*

$$E\left[\phi\left(X_\infty^*\right)\psi\left(\frac{1}{A_\infty}\right)\right] \leq E\left[(\phi + \Phi)(A_\infty)\,\psi\left(\frac{1}{A_\infty}\right) + \int_{\frac{1}{A_\infty}}^\infty \Phi\left(\frac{1}{y}\right) d\psi(y)\right]. \tag{5}$$

In particular, taking $\phi(x) = x^p$, $\psi(x) = x^q (0 \leq q < p < 1)$, then

$$E\left[\left(X_\infty^*\right)^p / A_\infty^q\right] \leq \left(\frac{p}{p-q} + \frac{1}{1-p}\right) E\left(A_\infty^{p-q}\right), \quad 0 \leq q < p < 1. \tag{6}$$

(3) *Let $\phi(x)$ be a right-continuous increasing function from R_+ to R_+, then*

$$E\left[\phi\left(X_\infty^*\right)\right] \leq E\left[(\phi + \Phi)(A_\infty)\right]. \tag{7}$$

In particular, taking $\phi(x) = x^p (0 < p < 1)$, one can obtain the Lenglart domination inequality (2)

$$E\left[\left(X_\infty^*\right)^p\right] \leq \frac{2-p}{1-p} E\left(A_\infty^p\right), \quad 0 < p < 1. \tag{8}$$

The aim of this paper is to present strengthened versions of the above domination inequalities.