Statistics and Probability Letters 81 (2011) 1871–1875

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Minimax lower bound for kink location estimators in a nonparametric regression model with long-range dependence

Justin Rory Wishart *

School of Mathematics & Statistics F07, The University of Sydney, NSW, 2006, Australia

ARTICLE INFO

Article history:
Received 4 April 2011
Received in revised form 26 July 2011
Accepted 28 July 2011
Available online 3 August 2011

MSC:
62G08
62G05
62G20

Keywords:
Nonparametric regression
Long-range dependence
Kink
Minimax

ABSTRACT

In this paper, a lower bound is determined in the minimax sense for change point estimators of the first derivative of a regression function in the fractional white noise model. Similar minimax results presented previously in the area focus on change points in the derivatives of a regression function in the white noise model or consider estimation of the regression function in the presence of correlated errors.

1. Introduction

Nonparametric estimation of a kink in a regression function has been considered for Gaussian white noise models by Cheng and Raimondo (2008), Goldenshluger et al. (2008a,b). Recently, this was extended to the fractional Gaussian noise model by Wishart (2009). The fractional Gaussian noise model assumes the regression structure

$$dY(x) = \mu(x)\,dx + \varepsilon^{\alpha}\,dB_H(x), \quad x \in \mathbb{R}, \tag{1}$$

where B_H is a fractional Brownian motion (fBm) and $\mu:\mathbb{R} \longrightarrow \mathbb{R}$ is the regression function. The level of error is controlled by $\varepsilon \asymp n^{-1/2}$, where the relation $a_n \asymp b_n$ means that the ratio a_n/b_n is bounded above and below by constants. The level of dependence in the error is controlled by the Hurst parameter $H \in (1/2, 1)$ and $\alpha := 2 - 2H$, where the i.i.d. model corresponds to $\alpha = 1$. The fractional Gaussian noise model was used by Johnstone and Silverman (1997) and Wishart (2009) among others to model regression problems with long-range dependent errors.

This study is interested in the performance of estimators of a change point in the first derivative of μ observed in model (1). This type of change point is called a kink, and the location is denoted by θ. Let $\widehat{\theta}_n$ denote an estimator of θ given n observations. A lower bound is established for the minimax rate of kink location estimation using the quadratic loss in the sense that

$$\liminf_{n\to\infty} \inf_{\widehat{\theta}_n} \sup_{\mu\in\mathscr{F}_s(\theta)} \rho_n^{-2}\mathbb{E}\left|\widehat{\theta}_n - \theta\right|^2 \geq C \quad \text{for some constant } C > 0. \tag{2}$$

* Tel.: +61 2 9114 1283; fax: +61 2 9351 4534.
E-mail address: justin.wishart@sydney.edu.au.
URL: http://www.sydney.edu.au/science/maths/u/justinw.

0167-7152/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2011.07.019

The main quantity of interest in this lower bound is the rate, ρ_n. In (2), $\inf_{\widehat{\theta}_n}$ denotes the infimum over all possible estimators of θ. The class of functions under consideration for μ is denoted $\mathscr{F}_s(\theta)$, defined below.

Definition 1. Let $s \geq 2$ be an integer, and let $a \in \mathbb{R} \setminus \{0\}$. Then, we say that $\mu \in \mathscr{F}_s(\theta)$ if the following hold.

1. The function μ has a kink at $\theta \in (0, 1)$. That is,
$$\lim_{x \downarrow \theta} \mu^{(1)}(x) - \lim_{x \uparrow \theta} \mu^{(1)}(x) = a \neq 0.$$
2. The function $\mu \in \mathscr{L}_2(\mathbb{R}) \cap \mathscr{L}_1(\mathbb{R})$, and satisfies the following condition:
$$\int_{\mathbb{R}} |\widetilde{\mu}(\omega)| \, |\omega|^s \, \mathrm{d}\omega < \infty, \tag{3}$$
where $\widetilde{\mu}(\omega) := \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mu(x) \, \mathrm{d}x$ is the Fourier transform of μ.

The minimax rate for the kink estimators has been discussed in the i.i.d. scenario by Cheng and Raimondo (2008) and Goldenshluger et al. (2008a), and was shown to be $n^{-s/(2s+1)}$. An extension of the kink estimators to the long-range dependent scenario was considered in Wishart (2009), which built on the work of Cheng and Raimondo (2008). An estimator of kink locations was constructed by Wishart (2009), and it achieved the rate, in the probabilistic sense,
$$\left|\widehat{\theta}_n - \theta\right| = \mathcal{O}_p(n^{-\alpha s/(2s+\alpha)}), \tag{4}$$
which includes the result of Cheng and Raimondo (2008) as a special case with the choice $\alpha = 1$. Both Cheng and Raimondo (2008) and Wishart (2009) considered a comparable model in the indirect framework, and used the results of Goldenshluger et al. (2006) to infer the minimax optimality of (4). However, the results of Cheng and Raimondo (2008) and Wishart (2009) require a slightly more restrictive functional class than $\mathscr{F}_s(\theta)$. The rate $n^{-s/(2s+1)}$, obtained by Cheng and Raimondo (2008), was confirmed as the minimax rate by the work of Goldenshluger et al. (2008a), who used the i.i.d. framework and a functional class similar to $\mathscr{F}_s(\theta)$.

The fBm concept is an extension of Brownian motion that can exhibit dependence among its increments which is typically controlled by the Hurst parameter, H (see Beran, 1994; Doukhan et al., 2003 for more detailed treatment on long-range dependence and fBm). The fBm process is defined below.

Definition 2. The fractional Brownian motion $\{B_H(t)\}_{t\in\mathbb{R}}$ is a Gaussian process with mean zero and covariance structure:
$$\mathbb{E}B_H(t)B_H(s) = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}.$$
We assume throughout the paper that $H \in (1/2, 1)$, whereby the increments of B_H are positively correlated and are long-range dependent.

In this paper, a lower bound for the minimax convergence rate of kink estimation using the quadratic loss function will be shown explicitly on model (1). This is a stronger result in terms of a lower bound than the simple probabilistic result in (4) given by Wishart (2009), and is applicable to a broader class of functions.

2. Lower bound

The aim of the paper is to establish the following result.

Theorem 1. *Suppose that $\mu \in \mathscr{F}_s(\theta)$ is observed from model (1) and that $0 < \alpha < 1$. Then, there exists a positive constant $C < \infty$ that does not depend on n such that the lower rate of convergence for an estimator for the kink location θ with the square loss is of the form*
$$\liminf_{n\to\infty} \inf_{\widehat{\theta}_n} \sup_{\mu\in\mathscr{F}_s(\theta)} n^{2\alpha s/(2s+\alpha)} \mathbb{E}\left|\widehat{\theta}_n - \theta\right|^2 \geq C.$$

From Theorem 1, one can see that the minimax rate for kink estimation in the i.i.d. case is recovered with the choice $\alpha = 1$ (see Goldenshluger et al., 2008a). Also unsurprisingly, the level of dependence is detrimental to the rate of convergence. For instance, as the increments become more correlated, and $\alpha \to 0$, the rate of convergence diminishes.

As will become evident in the proof of Theorem 1, the Kullback–Leibler divergence is required between two measures involving modified fractional Brownian motions. To cater for this, some auxiliary definitions to precede the proof of Theorem 1 are given in the next section.

3. Preliminaries

In this paper, the functions under consideration are defined in the Fourier domain (see Definition 1). Among others, there are two representations for fBm that satisfy Definition 2 that are used in this paper. The first is the moving average representation of Mandelbrot and Van Ness (1968) in the time domain and the second is the spectral representation given by Samorodnitsky and Taqqu (1994) in the Fourier domain. These both need to be considered since they are both used in