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Some inequalities for absolute moments

N.G. Ushakov

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491, Trondheim, Norway

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ABSTRACT

In this note, we obtain several inequalities for absolute moments of sums and differences of independent random variables, using one representation of absolute moments in terms of the characteristic function and inequalities for characteristic functions.

1. Introduction

In this paper, we obtain several new moment inequalities for absolute moments of fractional order. Proofs of these inequalities are based on representations of absolute moments in terms of characteristic functions. There are several such representations, see Hsu (1951), Zolotarev (1957) and von Bahr (1965). In this work, we will use the following one.

Lemma 1 (*Von Bahr*)**.** *Let X be a random variable with the distribution function $F(x)$ and the characteristic function $\varphi(t)$. Suppose that $\mathrm{E}|X|^p < \infty$, where $p > 0$ and p is not an even integer. Denote $\alpha_k = \mathrm{E}X^k$, k nonnegative integer. Then*

$$\mathrm{E}|X|^p = C(p)\int_{-\infty}^{\infty} \frac{\Re\varphi(t) - \sum_{k=0}^{s}(-1)^k\alpha_{2k}t^{2k}/(2k)!}{|t|^{p+1}}\mathrm{d}t \tag{1}$$

where s is the integer part of $p/2$, and

$$C(p) = \frac{\Gamma(p+1)}{\pi}\cos\frac{(p+1)\pi}{2}. \tag{2}$$

It turns out that combining this representation with inequalities for characteristic functions (sometimes very elementary), one obtains interesting inequalities for moments, often surprising and unexpected (because, in particular, they hold for moments of some orders but do not for others).

2. Main results and proofs

In what follows, $C(p)$ denotes the quantity given by (2).

E-mail address: ushakov@math.ntnu.no.

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Theorem 1. *Let X and Y be independent and identically distributed random variables with a finite absolute moment of order p.*
1. *If $0 < p < 2$, then*

$$\mathrm{E}|X - Y|^p \leq \mathrm{E}|X + Y|^p \tag{3}$$

with the equality if and only if X and Y are symmetric about zero.
2. *If $2 < p < 4$ and $\mathrm{E}X = \mathrm{E}Y = 0$, then*

$$\mathrm{E}|X - Y|^p \geq \mathrm{E}|X + Y|^p \tag{4}$$

with the equality if and only if X and Y are symmetric about zero.

Inequality (3) is a special case of inequality (2.1) in Buja et al. (1994). Nevertheless we present (3) here for completeness, for comparison with inequality (4) and because the proof is completely different with that in Buja et al. (1994). It has to be pointed out also that the characterization of symmetry, given by the first part of the theorem, was obtained by Braverman (1985).

Proof. First note that evidently $\mathrm{E}(X - Y)^2 \leq \mathrm{E}(X + Y)^2$ and

$$\mathrm{E}(X - Y)^2 = \mathrm{E}(X + Y)^2 \quad \text{iff } \mathrm{E}X = \mathrm{E}Y = 0. \tag{5}$$

Denote the characteristic function of X and Y by $\varphi(t)$.

Let $0 < p < 2$. Then $C(p) < 0$ and, using Lemma 1, we obtain

$$\mathrm{E}|X - Y|^p = -C(p)\int_{-\infty}^{\infty} \frac{1}{|t|^{p+1}}(1 - |\varphi(t)|^2)\mathrm{d}t \leq -C(p)\int_{-\infty}^{\infty} \frac{1}{|t|^{p+1}}(1 - \Re\varphi^2(t))\mathrm{d}t = \mathrm{E}|X + Y|^p.$$

If X and Y are symmetric then evidently

$$\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y|^p. \tag{6}$$

Suppose now that (6) holds. Then

$$\int_{-\infty}^{\infty} \frac{1}{|t|^{p+1}}(1 - |\varphi(t)|^2)\mathrm{d}t = \int_{-\infty}^{\infty} \frac{1}{|t|^{p+1}}(1 - \Re\varphi^2(t))\mathrm{d}t$$

and therefore

$$(\Re\varphi(t))^2 + (\Im\varphi(t))^2 = |\varphi(t)|^2 = \Re\varphi^2(t) = (\Re\varphi(t))^2 - (\Im\varphi(t))^2$$

i.e. $\Im\varphi(t) \equiv 0$. But this means that X is symmetric about zero.

Let $2 < p < 4$ and $\mathrm{E}X = \mathrm{E}Y = 0$. Then $C(p) > 0$ and, using Lemma 1 and (5), we obtain

$$\begin{aligned}\mathrm{E}|X - Y|^p &= C(p)\int_{-\infty}^{\infty} \frac{1}{|t|^{p+1}}(|\varphi(t)|^2 - 1 + \mathrm{E}(X - Y)^2 \cdot t^2/2)\mathrm{d}t \\ &\geq C(p)\int_{-\infty}^{\infty} \frac{1}{|t|^{p+1}}(\Re\varphi^2(t) - 1 + \mathrm{E}(X + Y)^2 \cdot t^2/2)\mathrm{d}t = \mathrm{E}|X + Y|^p.\end{aligned}$$

Proof that the equality in (4) takes place if and only if X is symmetric is similar to that in the case $0 < p < 2$. □

If $p = 4$ and $\mathrm{E}X = \mathrm{E}Y = 0$ then evidently $\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y|^p$.

Theorem 2. *Let X and Y be independent symmetric about zero random variables, X' an independent of X random variable having the same distribution as X, Y' an independent of Y random variable having the same distribution as Y.*
1. *If $0 < p < 2$ then*

$$\mathrm{E}|X + X'|^p + \mathrm{E}|Y + Y'|^p \leq 2\mathrm{E}|X + Y|^p \tag{7}$$

with equality if and only if X and Y are identically distributed.
2. *If $2 < p < 4$ then*

$$\mathrm{E}|X + X'|^p + \mathrm{E}|Y + Y'|^p \geq 2\mathrm{E}|X + Y|^p \tag{8}$$

with equality if and only if X and Y are identically distributed.