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Stochastic integration with respect to the sub-fractional Brownian motion with $H \in (0, \frac{1}{2})$

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We define a stochastic integral with respect to sub-fractional Brownian motion S^H with index $H \in (0, \frac{1}{2})$ that extends the divergence integral from Malliavin calculus. For this extended divergence integral, we establish versions of the formulas of Itô and Tanaka that hold for all $H \in (0, \frac{1}{2})$.

1. Introduction

Recently, the long-range dependence property has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing and finance. The best known and most widely used process that exhibits the long-range dependence property is *fractional Brownian motion* (fBm in short). The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity and has stationary increments. It is impossible to list here all the contributors in previous topics. Some surveys and the complete literature could be found in Biagini et al. (2008), Hu (2005), Mishura (2008), Nualart (2006) and Cheridito and Nualart (2005). On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm are introduced. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments.

As an extension of Brownian motion, recently, Bojdecki et al. (2004a) introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fBm. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the *sub-fractional Brownian*

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motion. The so-called sub-fractional Brownian motion (sub-fBm in short) with index $H \in (0, 1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \geq 0\}$ with $S_0^H = 0$ and the covariance

$$C_H(t,s) \equiv E\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right] \tag{1.1}$$

for all $s, t \geq 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion B. S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H. As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to S^H (see, for example, Alòs et al., 2001). The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths), enlarges the scope of behavior of fBm and it may be useful in some applications. But its increments are not stationary and satisfy the following estimates:

$$[(2 - 2^{2H-1}) \wedge 1](t-s)^{2H} \leq E\left[\left(S_t^H - S_s^H\right)^2\right] \leq [(2 - 2^{2H-1}) \vee 1](t-s)^{2H}. \tag{1.2}$$

Thus, Kolmogorov's continuity criterion implies that sub-fBm is Hölder continuous of order γ for any $\gamma < H$ on any finite interval. Therefore, if u is a stochastic process with Hölder continuous trajectories of order $\beta > 1 - H$, then the path-wise Riemann–Stieltjes integral $\int_0^T u_t(\omega) dS_t^H(\omega)$ exists for all $T \geq 0$ (see Young, 1936). In particular, if $H > \frac{1}{2}$, the path-wise integral $\int_0^T f'(S_t^H) dS_t^H$ exists for all $f \in C^2(\mathbb{R})$, and

$$f(S_T^H) - f(0) = \int_0^T f'(S_t^H) dS_t^H.$$

However, when $H < \frac{1}{2}$, the path-wise Riemann–Stieltjes integral $\int_0^T f'(S_t^H) dS_t^H(\omega)$ does not exist. For $H = \frac{1}{2}$, the stochastic integral introduced by Itô (1944) has been proven to be a very fruitful approach and has led to the development of classical stochastic calculus. Nualart and Pardoux (1988) proved that the Itô stochastic integral coincides with the divergence operator on the Wiener space. Later, Yan et al. (2011) have used the divergence operator to define the stochastic integrals with respect to sub-fBm with $H > \frac{1}{2}$ and showed that if $f \in C^2(\mathbb{R})$ and $\frac{1}{2} < H < 1$, then

$$f(S_t^H) = f(0) + \int_0^t f'(S_s^H) dS_s^H + H(2 - 2^{2H-1}) \int_0^t f''(S_s^H) s^{2H-1} ds.$$

More works for sub-fBm can be found in Bojdecki et al. (2004b, 2006, 2007), Dzhaparide and Van Zanten (2004), Liu and Yan (in press), Shen et al. (2011), Shen and Yan (2011) and Tudor (2007, 2008a,b, 2009), Yan et al. (2011), Yan and Shen (2011) and the references therein.

In this paper, our purpose is to define a stochastic integral with respect to sub-fBm with $H \in (0, \frac{1}{2})$ by using divergence operator from Malliavin calculus. The method used here is essentially due to Cheridito and Nualart (2005) and Coutin et al. (2001). It is organized as follows. In Section 2, we present some preliminaries for sub-fBm and discuss the first chaos of sub-fBm with index $H \in (0, \frac{1}{2})$. In Section 3, we prove that if $H \in (0, \frac{1}{4}]$, then the process $S_t^H 1_{(0,T]}(t)$ is not in the domain of the standard divergence operator. We then introduce an extended divergence operator. In Section 4, we give Itô and Tanaka formulas for sub-fBm with $H \in (0, \frac{1}{2})$.

2. The first chaos of sub-fBm with $H \in (0, \frac{1}{2})$

Let $(S_t^H, t \geq 0)$ be a sub-fBm with $0 < H < \frac{1}{2}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma\left\{S_t^H, t \geq 0\right\}$. We denote $\mathscr{E}_+$ the linear space of step functions

$$\mathscr{E}_+ = \left\{f: \mathbb{R}_+ \to \mathbb{R}: f = \sum_{i=1}^N a_i 1_{[t_i, t_{i+1})}: N \geq 1, 0 < t_1 < t_2 < \cdots < t_{N+1} < \infty, a_i \in \mathbb{R}\right\},$$

equipped with the inner product

$$\left\langle \sum_{i=1}^N a_i 1_{(t_i, t_{i+1}]}, \sum_{j=1}^N a_j 1_{(s_j, s_{j+1}]} \right\rangle_{\mathscr{E}_+} := \mathbb{E}\left[\sum_{i=1}^N a_i (S_{t_{i+1}}^H - S_{t_i}^H) \sum_{j=1}^N a_j (S_{s_{j+1}}^H - S_{s_j}^H)\right].$$

Then, the linear map

$$\sum_{i=1}^N a_i 1_{(t_i, t_{i+1}]} \mapsto \sum_{i=1}^N a_i (S_{t_{i+1}}^H - S_{t_i}^H) \tag{2.1}$$

is an isometry between the inner product spaces $\mathscr{E}_+$ and span $\left\{S_t^H, t \geq 0\right\} \subset L^2(\Omega)$, where span denotes the linear span.

There exists a Hilbert space of functions which contains $\mathscr{E}_+$ as a dense subspace. To describe this Hilbert space, we need the following notions of fractional calculus. We refer the reader to Samko et al. (1993) for details about fractional integrals and derivatives.