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The uniform central limit theorem for the tent map

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ABSTRACT

This paper considers the asymptotic behaviors of the processes generated by the classical ergodic tent map that is defined on the unit interval. We get the uniform version of central limit theorem for the tent map by using the method of uniformly integrable entropy. An application to Kolmogorov–Smirnov type result is provided.

1. Introduction and the main results

The tent map is an iterated function forming a discrete-time dynamical system. The tent map demonstrates a chaotic dynamical behavior. In Bae et al. (2010a), we have developed the uniform laws of large numbers generated by the tent map.

The aim of our work is to develop the central limit theorem (CLT) and the uniform CLT for the process generated by the tent map by employing Ziegler (1997)'s idea of the uniformly integrable entropy method.

We begin with illustrating the tent map. Let $\Omega = [0, 1]$ be the sample space, $\mathcal{A}$ be the Borel sets and P be the Lebesgue measure. The tent map on the unit interval is defined by

$$\varphi(y) = \begin{cases} 2y, & \text{for } 0 \le y < \dfrac{1}{2} \\ 2(1-y), & \text{for } \dfrac{1}{2} \le y \le 1. \end{cases}$$

The tent map is an iterated function, in the shape of a tent. More specifically, if you plot $\varphi(y)$ versus y, it has two linear sections which rise to meet at $[1/2, 1]$—it looks like a tent.

The tent map φ preserves Lebesgue measure and is equivalent to a shift and flip map τ on $\{0, 1\}^{\{0,1,2,\dots\}}$:

$$\tau(\omega_0, \omega_1, \omega_2, \dots) = \begin{cases} (\omega_1, \omega_2, \dots) & \text{if } \omega_0 = 0 \\ (1-\omega_1, 1-\omega_2, \dots) & \text{if } \omega_0 = 1. \end{cases}$$

We can think of $(\omega_0, \omega_1, \omega_2, \dots) \in \{0, 1\}^{\{0,1,2,\dots\}}$ as a point y in the unit interval $[0, 1]$ by putting $y = \sum_{i=0}^{\infty} \frac{\omega_i}{2^{i+1}}$. It is known that the map φ is ergodic. See Durrett (1991, p. 302).

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We now consider a series of stationary processes generated by the tent map φ.

First, we start with $f_{\frac{1}{2}}(y) = 1_{[0,\frac{1}{2})}(y)$. Then $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables which have uniform distribution with

$$P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 0) = \frac{1}{2}$$
$$P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 1) = \frac{1}{2}.$$

Therefore $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ is a sequence of stationary random variables. Observe that $Ef_{\frac{1}{2}}(y) = \frac{1}{2}$ and $\mathrm{Var}(f_{\frac{1}{2}}(y)) = \frac{1}{4}$. Define

$$T_n(1, 1) = n^{-1/2} \sum_{m=1}^{n} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right).$$

Then, by the central limit theorem for stationary processes (see Gordin (1969)) $T_n(1, 1)$ converges in distribution to a standard normal random variable.

Second, for fixed $j \in \mathbb{N}$ and for fixed $i = 1, 2, \ldots, 2^j$, we look at $f_{i,j}(y) = 1_{[\frac{i-1}{2^j},\frac{i}{2^j})}(y)$. Then $\{f_{i,j}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables with

$$P(f_{i,j}(\varphi^{m-1}(y)) = 0) = 1 - 2^{-j}$$
$$P(f_{i,j}(\varphi^{m-1}(y)) = 1) = 2^{-j}.$$

Observe that $Ef_{i,j}(y) = \frac{1}{2^j}$ and $\mathrm{Var}(f_{i,j}(y)) = \frac{1}{2^j}(1 - \frac{1}{2^j})$. Define

$$T_n(i, j) = n^{-1/2} \sum_{m=1}^{n} \frac{f_{i,j}(\varphi^{m-1}(y)) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}} \quad \text{for given } i \text{ and } j.$$

Then, similar as above, $T_n(i, j)$ converges in distribution to a standard normal random variable.

Third, for each fixed $j \in \mathbb{N}$, we consider the sum

$$f_j(y) = \sum_{i=1}^{2^j} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}}$$

of the random variables

$$\frac{f_{1,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}}, \frac{f_{2,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}}, \ldots, \frac{f_{2^j,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}}.$$

Then, for fixed $j \in \mathbb{N}$, being a sequence of identically distributed random variables, $\{f_j(\varphi^{m-1}(y)) : m \geq 1\}$ is stationary and ergodic process. Consider the equation

$$\sum_{i=1}^{2^j} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}} - \sum_{i=1}^{2^j-1} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}} = \frac{f_{2^j,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}}.$$

We simply denote

$$d^j(y) := \frac{f_{2^j,j}(y) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}}.$$

Observe that $d^j(y)$ is a uniformly bounded in the sense that $\sup_{j\in\mathbb{N}} |d^j(y)| < \infty$. Observe also that for $j, k \in \mathbb{N}$,

$$\mathrm{Cov}(d^j(y), d^k(y)) = \frac{2^{-j} \wedge 2^{-k} - 2^{-j} \cdot 2^{-k}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}\{2^{-k}(1 - 2^{-k})\}^{1/2}}$$

where $x \wedge y$ denotes the minimum of x and y. In Bae et al. (2010a), we have proven that

$$\sup_{j\in\mathbb{N}} \left| n^{-1} \sum_{m=1}^{n} d^j(\varphi^{m-1}(y)) \right| \to 0 \quad \text{almost surely and in the mean.} \tag{1}$$

Recall that $\Omega = [0, 1]$ is the sample space, $\mathcal{A}$ is the Borel sets and P is the Lebesgue measure. Then $\varphi : \Omega \to \Omega$ is a P-preserving measurable transformation. Assume that $\mathcal{F}_0 := \{\emptyset, \Omega\}$ is the φ-invariant σ-field (i.e. $\varphi^{-1}\mathcal{F}_0 \subset \mathcal{F}_0$),