Statistics and Probability Letters 81 (2011) 1458–1462

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

An elementary proof of the L^1 log-Sobolev inequality for Poisson point processes[☆]

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ARTICLE INFO

Article history:
Received 3 January 2011
Received in revised form 25 April 2011
Accepted 27 April 2011
Available online 4 May 2011

MSC:
60J75
60H30

Keywords:
Log-Sobolev inequality
Poisson point process
Weak Poincaré inequality

ABSTRACT

In this note we provide a new proof of the L^1 log-Sobolev inequality on the path space of Poisson point processes. Our proof is elementary in the sense that it avoids the use of the martingale representation on Poisson spaces. Moreover, the weak Poincaré inequality for the weighted Dirichlet form is presented.

1. Introduction

Consider a Poisson point process $X = \{X_t : t \geq 0\}$ on $\mathbb{R}^d$ with $X_0 = 0$ and the Lévy measure ν, which satisfies $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \left(|z|^2 \wedge 1\right) \nu(\mathrm{d}z) < \infty.$$

Let Λ be the distribution of X, which is a probability measure on the path space

$$W := \left\{ \sum_{i=1}^{\infty} x_i 1_{[t_i,\infty)} : i \in \mathbb{N}, x_i \in \mathbb{R}^d \setminus \{0\}, 0 \leq t_i \uparrow \infty \text{ as } i \uparrow \infty \right\}$$

equipped with the σ-algebra induced by $\{w \mapsto w_t : t \geq 0\}$.

According to Wu (2000) (see also Deng and Wang (2010)),

$$F = \tilde{F} \quad \Lambda\text{-a.e. implies that } F(w + x1_{[t,\infty)}) = \tilde{F}(w + x1_{[t,\infty)})$$
$$\text{for } (\Lambda \times \nu \times \mathrm{d}t)\text{-a.e. } (w, x, t) \in W \times \mathbb{R}^d \times [0, \infty). \tag{1}$$

☆ Supported in part by NNSFC (10721091), the 973-Project (2006CB805901), and the Excellent Doctoral Dissertation Foundation of Beijing Normal University.

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doi:10.1016/j.spl.2011.04.020

Then the quadric form

$$\mathscr{E}(F,G) := \int_{W\times\mathbb{R}^d\times[0,\infty)} \left[F(w+x1_{[t,\infty)})-F(w)\right]\left[G(w+x1_{[t,\infty)})-G(w)\right] \Lambda(\mathrm{d}w)\nu(\mathrm{d}x)\mathrm{d}t$$

is well-defined on

$$\mathscr{D}(\mathscr{E}) := \{F \in L^2(\Lambda) : \mathscr{E}(F,F) < \infty\};$$

that is, the value of $\mathscr{E}(F,G)$ does not depend on Λ-versions of F and G.

It is known that the Boltzmann–Shannon (for short: B–S) entropy of a nonnegative measurable function F w.r.t. the probability measure Λ is defined as

$$\mathrm{Ent}_\Lambda(F) = \Lambda(F\log F) - \Lambda(F)\log\Lambda(F),$$

where $0\log 0$ is understood as $\lim_{x\downarrow 0} x\log x = 0$. B–S entropy appears in many areas of science, especially in physics. Suppose that

$$\widetilde{\mathscr{D}}(\mathscr{E}) = \{F \in \mathscr{D}(\mathscr{E}) : F \geq 0, \mathscr{E}(F,\log F) < \infty\}.$$

Theorem 1.1. *We have*

$$\mathrm{Ent}_\Lambda(F) \leq \mathscr{E}(F,\log F), \quad F \in \widetilde{\mathscr{D}}(\mathscr{E}).$$

The above inequality, called the L^1 log-Sobolev inequality or relative entropy inequality, was established in Proposition 4.1 of Wu (2000) by exploring the martingale representation (the counterpart on Poisson space of the Clark–Ocône formula over Wiener space). More precisely, it was proved in Wu (2000) that

$$\begin{aligned}\mathrm{Ent}_\Lambda(F) \leq &\int_{W\times\mathbb{R}^d\times[0,\infty)} \Big\{F(w+x1_{[t,\infty)})\left[\log F(w+x1_{[t,\infty)}) - \log F(w)\right] \\ &- \left[F(w+x1_{[t,\infty)})-F(w)\right]\Big\} \Lambda(\mathrm{d}w)\nu(\mathrm{d}x)\mathrm{d}t \\ \leq &\ \mathscr{E}(F,\log F), \quad F \in \widetilde{\mathscr{D}}(\mathscr{E}),\end{aligned}$$

where the second inequality is due to the elementary inequality

$$a(\log a - \log b) - (a-b) \leq (a-b)(\log a - \log b), \quad a, b \geq 0.$$

The purpose of this note is to give a new proof of the L^1 log-Sobolev inequality for $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. The proof relies only on semigroup calculus, the Markov property and the induction argument. Therefore, our proof is more elementary since it avoids the use of the martingale representation on Poisson spaces and the Itô formula for jump processes. We remark that the idea of our proof is essentially due to Wang and Yuan (2010).

The remainder of the note is organized as follows. In Section 2, we present the elementary proof of Theorem 1.1. Finally, we investigate the weak Poincaré inequality for weighted Dirichlet forms in Section 3.

2. The elementary proof of Theorem 1.1

We shall adopt an induction argument as in Wang and Yuan (2010). By the monotone class theorem and an approximation argument, we shall assume that

$$F(w) = f(w_{t_1}, \ldots, w_{t_n})$$

for $0 = t_0 < t_1 < \cdots < t_n$ and some $f \in C_b\left((\mathbb{R}^d)^n\right)$ with $0 < \inf f \leq \sup f < \infty$, $n \in \mathbb{N}$. Then the desired L^1 log-Sobolev inequality follows from

$$\begin{aligned}&\mathbb{E}^z\left[f(X_{t_1},\ldots,X_{t_n})\log f(X_{t_1},\ldots,X_{t_n})\right] - \mathbb{E}^z f(X_{t_1},\ldots,X_{t_n})\log\mathbb{E}^z f(X_{t_1},\ldots,X_{t_n}) \\ &\quad\leq \sum_{i=1}^n (t_i - t_{i-1})\mathbb{E}^z \int_{\mathbb{R}^d} \left[f(X_{t_1},\ldots,X_{t_{i-1}},X_{t_i}+x,\ldots,X_{t_n}+x) - f(X_{t_1},\ldots,X_{t_n})\right] \\ &\quad\quad\times\log\frac{f(X_{t_1},\ldots,X_{t_{i-1}},X_{t_i}+x,\ldots,X_{t_n}+x)}{f(X_{t_1},\ldots,X_{t_n})}\nu(\mathrm{d}x), \quad z \in \mathbb{R}^d,\end{aligned} \tag{2}$$

where $\mathbb{E}^z$ is the expectation taken for the Poisson point process starting at point z. We shall prove this inequality by iterating in n.

(a) Suppose that $n = 1$ and $t_1 = t > 0$. Then (2) reduces to

$$\mathbb{E}^z\left[f(X_t)\log f(X_t)\right] - \mathbb{E}^z f(X_t)\log\mathbb{E}^z f(X_t) \leq t\mathbb{E}^z \int_{\mathbb{R}^d} \left[f(X_t+x) - f(X_t)\right]\log\frac{f(X_t+x)}{f(X_t)}\nu(\mathrm{d}x). \tag{3}$$