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The beta Laplace distribution

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ABSTRACT

The Laplace distribution is one of the earliest distributions in probability theory. For the first time, based on this distribution, we propose the so-called beta Laplace distribution, which extends the Laplace distribution. Various structural properties of the new distribution are derived, including expansions for its moments, moment generating function, moments of the order statistics, and so forth. We discuss maximum likelihood estimation of the model parameters and derive the observed information matrix. The usefulness of the new model is illustrated by means of a real data set.

1. Introduction

One of the earliest distributions in probability theory was introduced by Laplace in 1774 (Laplace, 1774). A random variable Z has the Laplace distribution with location parameter μ and scale parameter $\sigma > 0$, say $Z \sim \mathrm{L}(\mu, \sigma)$, if its probability density function (pdf) is given by

$$g(z) = \frac{1}{2\sigma} \exp\left\{-\frac{|z-\mu|}{\sigma}\right\}, \quad -\infty < z < \infty.$$

The mean, median and mode are all equal to μ. The variance is $2\sigma^2$ and the skewness and kurtosis are 0 and 6, respectively. The moment generating function (mgf) of Z is $M(t) = (1+\sigma^2 t^2)^{-1}\exp(\mu t)$. In addition, the cumulative distribution function (cdf) becomes

$$G(z) = \begin{cases} \dfrac{1}{2}\exp\left(\dfrac{z-\mu}{\sigma}\right), & z < \mu, \\ 1 - \dfrac{1}{2}\exp\left(-\dfrac{z-\mu}{\sigma}\right), & z \geq \mu. \end{cases}$$

If we consider the standardized random variable $X = (Z-\mu)/\sigma$, the pdf of X reduces to $g(x) = \frac{1}{2}\exp(-|x|)$, $-\infty < x < \infty$, and the corresponding cdf and mgf are given by

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$$G(x) = \begin{cases} \frac{1}{2}\exp(x), & x < 0, \\ 1 - \frac{1}{2}\exp(-x), & x \geq 0, \end{cases}$$

and $M(t) = (1 + t^2)^{-1}$, respectively. In this case, $X \sim \mathrm{L}(0, 1)$.

The Laplace distribution, also named the double exponential distribution, and its variants are becoming popular in many areas of science and engineering. This distribution is often used for modeling phenomena with "heavier than normal tails"; see for example, Andrews et al. (1972), Manly (1976), Easterling (1978), Hsu (1979), Bagchi et al. (1983), Hoaglin et al. (1983), Dadi and Marks (1987), Damsleth and El–Shaarawi (1989), Puig and Stephens (2000), Chen (2002), and also Johnson et al. (1995) which contains a detailed list of references. A book-length account of Laplace distributions, discussing in great detail their various properties and applications, is available due to Kotz et al. (2001).

In this article we propose a new model, so-called the beta Laplace (BL) distribution, which contains as a sub-model the Laplace distribution. The BL distribution is convenient for modeling asymmetric data as a competitive model to beta normal and skew-normal distributions. We obtain some mathematical properties, discuss maximum likelihood estimation of the parameters and derive the observed information matrix. The article is outlined as follows. In Section 2, we introduce the BL distribution and provide plots of the density function. We demonstrate that the BL density function can be expressed as an infinite linear combination of Laplace density functions in Section 3. We provide in Section 4 a general expansion for the moments and mgf. Expansions for the quantile function and mean deviations are provided in Section 5. In Section 6, we demonstrate that the density function of the BL order statistics can be written as a linear combination of Laplace densities. We also obtain expansions for the moments of the order statistics. The Rényi and Shannon entropies are derived in Section 7. Maximum likelihood estimation is addressed in Section 8. Section 9 illustrates the importance of the BL distribution through the analysis of a real data set. Finally, Section 10 offers some concluding remarks.

2. The beta Laplace distribution

The generalization of the Laplace distribution is motivated by the work of Eugene et al. (2002) who defined a class of generalized beta distributions by

$$F(x) = \frac{1}{B(a, b)} \int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1}\mathrm{d}\omega = I_{G(x)}(a, b). \tag{1}$$

Here, $a > 0$ and $b > 0$ are two additional parameters which control skewness through the relative tail weights, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y \omega^{a-1}(1-\omega)^{b-1}\mathrm{d}\omega$ is the incomplete beta function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$ is the gamma function. This class of generalized distributions has been receiving considerable attention over the last years in particular after the work of Jones (2004). The probability density function (pdf) corresponding to (1) is $f(x) = g(x)G(x)^{a-1}\{1 - G(x)\}^{b-1}/B(a, b)$, where $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ is the parent density function. The density $f(x)$ will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions. Except for special choices of these functions, $f(x)$ will be difficult to deal with some generality.

Eugene et al. (2002), Nadarajah and Gupta (2004), Nadarajah and Kotz (2004, 2006), Lee et al. (2007) and Akinsete et al. (2008) defined the beta normal, beta Fréchet, beta Gumbel, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cdf of the normal, Fréchet, Gumbel, exponential, Weibull and Pareto distributions, respectively. More recently, Barreto-Souza et al. (2010), Pescim et al. (2010) and Cordeiro and Lemonte (2011) introduced the beta generalized exponential, the beta generalized half-normal and the beta Birnbaum–Saunders distributions, respectively.

The cdf of the BL distribution can be written as

$$F(x) = \begin{cases} I_{\exp(x)/2}(a, b), & x < 0, \\ I_{1-\exp(-x)/2}(a, b), & x \geq 0. \end{cases} \tag{2}$$

The density function corresponding to (2) is given by

$$f(x) = \begin{cases} \{2^a B(a, b)\}^{-1} \exp(-|x|) \exp\{-|x|(a-1)\}\{1 - \exp(-|x|)/2\}^{b-1}, & x < 0, \\ \{2^b B(a, b)\}^{-1} \exp(-|x|) \exp\{-|x|(b-1)\}\{1 - \exp(-|x|)/2\}^{a-1}, & x \geq 0. \end{cases} \tag{3}$$

We note that the case $x < 0$ can be obtained from the case $x \geq 0$ by replacing x for $-x$ and interchanging a and b. Clearly, for $a = b = 1$, Eq. (3) reduces to the standard Laplace density function. If X follows (3), we write $X \sim \mathrm{BL}(a, b)$. Plots of the $\mathrm{BL}(a, b)$ distribution are illustrated in Fig. 1 for selected parameter values, including the special case of the standard Laplace distribution. It is evident that the BL distribution is much more flexible than the Laplace distribution.

The properties of a random variable Z having the BL distribution with location parameter μ and dispersion parameter σ, say $Z \sim \mathrm{BL}(\mu, \sigma, a, b)$, can be determined directly from those properties of X using the linear transformation $Z = \mu + \sigma X$.

3. Expansions

First, if $|z| < 1$ and $b > 0$ is real non-integer, we have the power series expansion $(1-z)^{b-1} = \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j z^j$. Applying this expansion for $x < 0$, we obtain