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On the solution process for a stochastic fractional partial differential equation driven by space–time white noise

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ABSTRACT

Let $\{u(t,x): t \geq 0, x \in \mathbb{R}\}$ be the solution process for the following Cauchy problem for the stochastic fractional partial differential equation taking values in $\mathbb{R}^d$:

$$\frac{\partial}{\partial t}u(t,x) = \mathfrak{D}_\alpha^\delta u(t,x) + \dot{W}(t,x), \quad t > 0,\ x \in \mathbb{R}; \qquad u(0,x) = u^0(x),$$

where $\mathfrak{D}_\alpha^\delta$ $(1 < \alpha < 3, |\delta| \leq \min\{\alpha - [\alpha], 2 + [\alpha]_2 - \alpha\})$ is the fractional differential operator with respect to the spatial variable x (see below for a definition), $\dot{W}(t,x)$ is an $\mathbb{R}^d$-valued space–time white noise, and u^0 is an initial random datum defined on $\mathbb{R}$.

In this paper, we study the sample path properties of the solution process. We first find the dimensions in which the process hits points, and then determine the Hausdorff and packing dimensions of the range, the graph and the level sets of the process. Our results generalize those of Mueller and Tribe (2002) and Wu and Xiao (2006) for random string processes.

1. Introduction

Fractional equations have received more attention in recent years. Various phenomena have been modeled by means of fractional equations in physics, image analysis, and risk management, among other areas (cf. Le Mehaute et al. (2004) and Uchaikin and Zolotarev (1999) for a survey of applications). Several researchers have studied various problems on fractional equations; see Debbi and Dozzi (2005) and references therein. Very recently, Niu and Xie (2010) gave a deep study of the work introduced by Debbi and Dozzi (2005), where they dealt with one-dimensional nonlinear stochastic fractional differential equations driven by space–time white noise, and investigated the existence, uniqueness and regularities of the solutions.

Following Debbi and Dozzi (2005), a differential operator $\mathfrak{D}_\alpha^\delta$ is called a *fractional differential operator* with parameters $\alpha,\ \delta$ if $\mathfrak{D}_\alpha^\delta$ is defined by

$$\mathfrak{D}_\alpha^\delta \phi(x) := \mathfrak{F}^{-1}\left\{-|\cdot|^\alpha e^{-i\delta\pi\,\mathrm{sgn}(\cdot)/2}\mathfrak{F}(\phi)\right\}(x), \tag{1.1}$$

where $\alpha \in (0,\ \infty)$, $\delta \leq \min\{\alpha - [\alpha],\ 2 + [\alpha]_2 - \alpha\}$, $[\alpha]$ and $[\alpha]_2$ denote the largest integer and the largest even integer no greater than α, and $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are the Fourier and Fourier inverse transformations respectively, defined by

$$\mathfrak{F}\{\phi\}(\lambda) := \widehat{\phi}(\lambda) = \int_{-\infty}^{\infty} \exp(ix\lambda)\phi(x)\,dx,$$

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$$\mathfrak{F}^{-1}\{\widehat{\phi}\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}x\lambda)\widehat{\phi}(\lambda)\,\mathrm{d}\lambda.$$

This operator is a closed densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. It is a generalization of various well-known operators, such as the Laplacian operator (when $\alpha = 2$), the inverse of the generalized Riesz–Feller potential (when $\alpha > 2$) and the Riemann–Liouville operator (when $|\delta| = \alpha - [\alpha]$, or $2 + [\alpha]_2 - \alpha$). We refer the reader to Debbi (2007) and Debbi and Dozzi (2005) for more details about this operator.

Consider the Cauchy problem of the stochastic fractional partial differential equation, taking values in $\mathbb{R}^d$:

$$\frac{\partial}{\partial t}u(t,x) = \mathfrak{D}_\alpha^\delta u(t,x) + \dot{W}(t,x), \quad t > 0,\ x \in \mathbb{R}; \tag{1.2}$$

$$u(0,x) = u^0(x), \tag{1.3}$$

where $\mathfrak{D}_\alpha^\delta$ $(1 < \alpha < 3,\ |\delta| \le \min\{\alpha - [\alpha],\ 2 + [\alpha]_2 - \alpha\})$ is the fractional differential operator with respect to the spatial variable x defined by (1.1), and $\dot{W}(t,x)$ is a space–time white noise in $\mathbb{R}^d$.

Let the Green function $G_\alpha(t,x)$ associated with (1.2) be the fundamental solution of the Cauchy problem

$$\frac{\partial}{\partial t}G(t,x) = \mathfrak{D}_\alpha^\delta G(t,x), \quad t > 0,\ x \in \mathbb{R}; \qquad G(0,x) = \delta_0(x), \tag{1.4}$$

where δ_0 denotes the Dirac function at the point 0. Then $G_\alpha(t,x)$ can be expressed using Fourier analysis:

$$G_\alpha(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\mathrm{i}\xi x - t|\xi|^\alpha \mathrm{e}^{-\mathrm{i}\delta\pi\,\mathrm{sgn}(\xi)/2}\right\} \mathrm{d}\xi. \tag{1.5}$$

The function $G_\alpha(t,x)$ has the following properties (cf. Debbi and Dozzi (2005)):

(i) $\int_{-\infty}^{\infty} G_\alpha(t,x)\,\mathrm{d}x = 1$.
(ii) $G_\alpha(t,x)$ is real, but it is not symmetric relative to x and it is not everywhere positive in general.
(iii) $G_\alpha(t,x)$ satisfies the semigroup property, that is, for $0 < s < t$

$$G_\alpha(t+s,x) = \int_{-\infty}^{\infty} G_\alpha(t,y)G_\alpha(s,x-y)\,\mathrm{d}y.$$

(iv) For $0 < \alpha \le 2$, the function $G_\alpha(t,\cdot)$ is the density of a Lévy stable process in time t.
(v) For fixed t, $G_\alpha(t,\cdot) \in S^\infty$, where

$$S^\infty = \left\{ f \in C^\infty : \frac{\partial^\beta f}{\partial x^\beta} \text{ are bounded and tend to 0 as } |x| \to \infty,\ \forall \beta \in \mathbb{R}_+ \right\}.$$

(vi) $G_\alpha(t,x)$ has the following scaling property: $G_\alpha(t,x) = t^{-1/\alpha}G_\alpha(1, t^{-1/\alpha}x)$.
(vii) For $\alpha \in (1, \infty) \setminus \mathbb{N}$, there exists a constant K_α such that

$$|G_\alpha(1,x)| \le K_\alpha \left(1 + |x|^{1+\alpha}\right)^{-1}.$$

(viii) For $1 < \gamma < \alpha + 1$,

$$\int_0^\infty \int_{-\infty}^\infty |G_\alpha(1+s,z) - G_\alpha(s,z)|^\gamma \ \mathrm{d}z\,\mathrm{d}s < \infty.$$

(ix) For $\frac{\alpha+1}{2} < \gamma < \alpha + 1$,

$$\int_0^\infty \int_{-\infty}^\infty |G_\alpha(s,1+z) - G_\alpha(s,z)|^\gamma \ \mathrm{d}z\,\mathrm{d}s < \infty.$$

We assume that $\{\dot{W}(t,x)\}$ is adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}$ is complete and the filtration $\{\mathcal{F}_t, t \ge 0\}$ is right continuous. The components $\dot{W}_1(x,t), \ldots, \dot{W}_d(x,t)$ of $\dot{W}(x,t)$ are independent space–time white noises, which are generalized Gaussian processes with covariance given by

$$\mathbb{E}\big[\dot{W}_j(x,t)\dot{W}_j(y,s)\big] = \delta(x-y)\delta(t-s), \quad (j = 1, \ldots, d).$$

That is, for every $1 \le j \le d$, $W_j(f)$ is a random field indexed by functions $f \in L^2([0,\infty)\times\mathbb{R})$ and, for all $f, g \in L^2([0,\infty)\times\mathbb{R})$, we have

$$\mathbb{E}\big[W_j(f)W_j(g)\big] = \int_0^\infty \int_{\mathbb{R}} f(t,x)g(t,x)\,\mathrm{d}x\mathrm{d}t.$$