Statistics and Probability Letters 115 (2016) 45–53

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Representation of stationary and stationary increment processes via Langevin equation and self-similar processes

Lauri Viitasaari *

Department of Mathematics and System Analysis, Aalto University School of Science, Helsinki P.O. Box 11100, FIN-00076 Aalto, Finland
Department of Mathematics, Saarland University, Post-fach 151150, D-66041 Saarbrücken, Germany

ARTICLE INFO

Article history:
Received 21 May 2015
Accepted 30 March 2016
Available online 7 April 2016

MSC:
60G07
60G10
60G18

Keywords:
Stationary processes
Stationary increment processes
Self-similar processes
Lamperti transform
Langevin equation

ABSTRACT

Let W_t be a standard Brownian motion. It is well-known that the Langevin equation $\mathrm{d}U_t = -\theta U_t \mathrm{d}t + \mathrm{d}W_t$ defines a stationary process called Ornstein–Uhlenbeck process. Furthermore, Langevin equation can be used to construct other stationary processes by replacing Brownian motion W_t with some other process G with stationary increments. In this article we prove that the converse also holds and all continuous stationary processes arise from a Langevin equation with certain noise $G = G_\theta$. Discrete analogies of our results are given and applications are discussed.

1. Introduction

Let $G = (G_t)_{t\in\mathbb{R}}$ be a continuous process with stationary increments and consider a Langevin equation

$$\mathrm{d}U_t = -\theta U_t \mathrm{d}t + \mathrm{d}G_t, \quad t \in \mathbb{R} \tag{1.1}$$

with some condition on U_0. If $G = W$ is a standard Brownian motion, then it is well-known that the Eq. (1.1) has a stationary solution that is called the Ornstein–Uhlenbeck process. Consequently, Langevin equation is connected to Ornstein–Uhlenbeck processes and can be used to construct stationary processes. Furthermore, Langevin type equations also have applications in statistical physics which highlights the importance of these equations even more.

A natural question related to Eq. (1.1) is whether it has a stationary solution with suitably chosen initial condition for more general noise term G, and this question was studied recently in Barndorff-Nielsen and Basse-O'Connor (2011) for general stationary increment noise G. The main result in Barndorff-Nielsen and Basse-O'Connor (2011) was that under mild integrability assumptions on the driving force G, Eq. (1.1) has a stationary solution.

Another useful tool to construct stationary processes is via Lamperti theorem (Lamperti, 1962) which states that each H-self-similar process X (for details on self-similar processes we refer to monographs Embrechts and Maejima, 2002; Flandrin et al., 2003; Tudor, 2013 dedicated to the subject) can be written as a Lamperti-transform $X = \mathcal{L}_H Y$, where Y is a stationary process. Moreover, Lamperti-transform is invertible and hence stationary processes can be constructed from

* Correspondence to: Department of Mathematics, Saarland University, Post-fach 151150, D-66041 Saarbrücken, Germany.
E-mail address: lauri.viitasaari@aalto.fi.

http://dx.doi.org/10.1016/j.spl.2016.03.020
0167-7152/

H-self-similar process via inverse transform $Y = \mathcal{L}_H^{-1}X$. Self-similar processes also have numerous applications. For example, the connection between self-decomposable laws and Levy processes (for details, we refer to Bertoin, 1996; Sato, 1999) with self-similar processes is studied in Jurek and Vervaat (1983), Wolfe (1982), Sato (1991) and Jeanblanc et al. (2002) to name a few. Especially, Jeanblanc et al. (2002) used H-self-similar processes to connect two different representations derived in Jurek and Vervaat (1983), Wolfe (1982) and in Sato (1991) for self-decomposable laws where the first representation is in terms of Levy processes (so called background driving Levy process) and the second representation is in terms of H-self-similar processes.

A process with special interest is the case of fractional Brownian motion with $H \in (0,1)$ and Ornstein–Uhlenbeck processes associated with it has been studied in Bercu et al. (2011), Cheridito et al. (2003) and Kaarakka and Salminen (2011). Recall that B^H is the only Gaussian process which is H-self-similar and has stationary increments. Consequently, both approaches can be used to construct stationary processes. However, it is also known that the resulting processes are the same (in law) only in the case $H = \frac{1}{2}$, i.e. in the case of standard Brownian motion. On the other hand, it was proved by Kaarakka and Salminen (2011) that even the Lamperti transform of fractional Brownian motion can be defined as a solution to Langevin type equation with some driving noise G. Furthermore, statistical problems for fractional Ornstein–Uhlenbeck processes have been studied at least in Azmoodeh and Morlanes (2015), Azmoodeh and Viitasaari (in press), Brouste and Iacus (2012), Hu and Nualart (2010), Kleptsyna and Le Breton (2002) and Xiao et al. (2011). For research related to more general self-similar Gaussian processes, see also Nuzman and Poor (2011) and Yazigi (2015).

In this article we make use of the Lamperti theorem to characterise (continuous) stationary processes as solutions to the Langevin equation (1.1) with some noise process G belonging to a certain class $\mathcal{G}_H$. More precisely, as our main result we show that a process is stationary if and only if it is a solution to the Langevin equation with noise $G \in \mathcal{G}_H$. As a simple consequence it follows that Langevin equation (1.1) has a stationary solution if and only if the noise process G belongs to $\mathcal{G}_H$; a result which generalises the main findings of Barndorff-Nielsen and Basse-O'Connor (2011). Moreover, we characterise the class $\mathcal{G}_H$ in terms of H-self-similar processes, and hence the results of this paper connect the two mentioned approaches to construct stationary processes in a natural way. We also present discrete analogies to our main theorems and consider some applications. For example, as a consequence of our main result we obtain that all discrete time stationary models reduces to a $AR(1)$-*model* with a non-white noise.

The rest of the paper is organised as follows. In Section 2 we introduce our notation and preliminary results, and in Section 3 we present and prove our main results. Section 4 is devoted to applications and examples.

2. Notation and preliminaries

Throughout the paper we assume that all processes $(X_t)_{t\in\mathbb{R}}$ have continuous paths almost surely (for extensions, see Remark 2.1). Consequently, we can define integrals of form $\int_s^t e^{Hu}\mathrm{d}X_u$ over a compact interval $[s,t]$ and some constant $H \in \mathbb{R}$ via integration by parts formula

$$\int_s^t e^{Hu}\mathrm{d}X_u = e^{Ht}X_t - e^{Hs}X_s - H\int_s^t X_u e^{Hu}\mathrm{d}u,$$

where the last integral is understood as a Riemann integral. For numbers $t < s$ we use standard definition $\int_s^t = -\int_t^s$. We also consider indefinite integrals of type $\int_{-\infty}^t e^{Hu}\mathrm{d}X_u$ which are defined similarly as

$$\int_{-\infty}^t e^{Hu}\mathrm{d}X_u = e^{Ht}X_t - H\int_{-\infty}^t e^{Hu}X_u\mathrm{d}u$$

provided that the integral on the right exists almost surely.

Remark 2.1. We remark that for our purposes, the key ingredient is the fact that the above integrals under consideration can be defined as Riemann integrals and integration by parts is valid. Hence the assumption of almost sure continuity is not needed *a priori* but it is made to ensure that the integrals considered can be understood pathwise without any additional technicalities. In comparison, in Barndorff-Nielsen and Basse-O'Connor (2011) the authors assumed integrability of the driving force of the Langevin equation which ensured that the integrals can be understood as L^p-limits of Riemann–Stieltjes sums. For generalisations, see also Section 4.1.

We denote $X_t \overset{\text{law}}{=} Y_t$ if finite dimensional distributions of X and Y are equal. Throughout the paper, we consider strictly stationary processes, i.e. processes $U = (U_t)_{t\in\mathbb{R}}$ for which $U_{t+h} \overset{\text{law}}{=} U_t$ for every $h \in \mathbb{R}$. However, we remark that all the results of this paper are true for covariance stationary processes as well.

Next we recall definition of Lamperti transform and its inverse together with the famous Lamperti theorem. First we recall the definition of self-similar processes.

Definition 2.1. Let $H > 0$. A process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ is H-self-similar if

$$X_{at} \overset{\text{law}}{=} a^H X_t$$

for every $a > 0$. The class of all H-self-similar processes on $[0,\infty)$ are denoted by $\mathcal{X}_H$.