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Can the bounds in the multivariate Chebyshev inequality be attained?

Jorge Navarro*

Facultad de Matematicas, Universidad de Murcia, 30100 Murcia, Spain

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ABSTRACT

Chebyshev's inequality was recently extended to the multivariate case. In this paper we prove that the bounds in the multivariate Chebyshev's inequality for random vectors can be attained in the limit. Hence, these bounds are the best possible bounds for this kind of regions.

1. Introduction

The very well known Chebyshev's inequality for random variables provides a lower bound for the percentage of the population in a given distance with respect to the population mean when the variance is known. There are several extensions of this inequality to the multivariate case (see e.g. Chen (2011), Marshall and Olkin (1960) and the references therein).

Recently, Chen (2011) proved the following multivariate Chebyshev's inequality

$$\Pr((\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu)\geq\varepsilon)\leq\frac{k}{\varepsilon} \tag{1}$$

valid for all $\varepsilon>0$ and for all random vectors $\mathbf{X}=(X_1,\dots,X_k)'$ (w' denotes the transpose of w) with the finite mean vector $\mu=E(\mathbf{X})$ and the positive definite covariance matrix $V=Cov(\mathbf{X})=E((\mathbf{X}-\mu)(\mathbf{X}-\mu)')$. Of course, (1) can also be written as

$$\Pr((\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu)<\varepsilon)\geq 1-\frac{k}{\varepsilon}$$

for all $\varepsilon>0$ or as

$$\Pr(d_V(\mathbf{X},\mu)<\delta)\geq 1-\frac{k}{\delta^2} \tag{2}$$

for all $\delta>0$, where

$$d_V(\mathbf{X},\mu)=\sqrt{(\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu)}$$

is the Mahalanobis distance associated with V between $\mathbf{X}$ and μ. Therefore (2) provides a lower bound for the probability in the concentration ellipsoid

$$E_\delta=\{\mathbf{x}\in\mathbb{R}^k:d_V(\mathbf{x},\mu)<\delta\}.$$

* Tel.: +34 868883508; fax: +34 868884182.
E-mail address: jorgenav@um.es.

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A simple proof of (1) was obtained in Navarro (in press). The case of a singular covariance matrix is also studied in this reference by using the principal components associated with X. Extensions of (1) to Hilbert-space-valued and Banach-space-valued random elements can be seen in Prakasa Rao (2010) and Zhou and Hu (2012), respectively.

Budny (2014) extends the inequality given in (1) by the following inequality

$$\Pr((\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu)\geq\varepsilon)\leq\frac{I_{s,k}(\mathbf{X})}{\varepsilon^{s}} \tag{3}$$

whenever $I_{s,k}(\mathbf{X})=E[((\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu))^{s}]$ is finite (and known) for $s>0$. In particular, if $s=2$, then

$$\Pr((\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu)\geq\varepsilon)\leq\frac{I_{2,k}(\mathbf{X})}{\varepsilon^{2}}. \tag{4}$$

This inequality was obtained in Mardia (1970), where $I_{2,k}(\mathbf{X})=E[((\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu))^{2}]$ is presented as a multivariate kurtosis coefficient. If $s=1$, then the bound obtained from (3) is the same as that given in (1) since $I_{1,k}(\mathbf{X})=E[(\mathbf{X}-\mu)'V^{-1}(\mathbf{X}-\mu)]=k$ for all $\mathbf{X}$ (see Navarro, in press). Sometimes, the bounds given in (3) and (4) are better bounds than that given in (1) but note that there we need to know $I_{s,k}(\mathbf{X})$ for a $s>0$ ($s\neq 1$).

In this paper we prove that the bound in (1) can be attained in the limit for all $\varepsilon\geq k$. Hence, this bound cannot be improved (for this kind of regions) when we only know μ and V. The proof is given in the next section and it is based on the proof of (1) given in Navarro (in press). The same technique is also used to prove that the bounds given in (3) can also be attained in the limit when $s>0$. Some conclusions are given in the last section.

2. Main results

The following theorem proves that the bound in (1) can be attained in the limit.

Theorem 1. *Let $\mathbf{X}=(X_1,\dots,X_k)'$ be a random vector with the finite mean vector $\mu=E(\mathbf{X})$ and the positive definite covariance matrix $V=Cov(\mathbf{X})$ and let $\varepsilon\geq k$. Then there exists a sequence $\mathbf{X}^{(n)}=(X_1^{(n)},\dots,X_k^{(n)})'$ of random vectors with the mean vector μ and the covariance matrix V such that*

$$\lim_{n\to\infty}\Pr((\mathbf{X}^{(n)}-\mu)'V^{-1}(\mathbf{X}^{(n)}-\mu)\geq\varepsilon)=\frac{k}{\varepsilon}. \tag{5}$$

Proof. For a fixed $\varepsilon\geq k$, let us consider the random variable D_n defined by

$$D_n=\begin{cases}\sqrt{Z_n+\varepsilon} & \text{with probability } (p-1/n)/2\\ -\sqrt{Z_n+\varepsilon} & \text{with probability } (p-1/n)/2\\ 0 & \text{with probability } 1-p+1/n\end{cases}$$

for any positive integer $n>\varepsilon/k$, where $p=k/\varepsilon\leq 1$ and where Z_n has an exponential distribution with mean

$$\mu_n=\frac{\varepsilon/n}{p-1/n}>0.$$

Note that

$$p-\frac{1}{n}=\frac{k}{\varepsilon}-\frac{1}{n}\in(0,1).$$

Also note that

$$\Pr(D_n^2\geq\varepsilon)=p-1/n.$$

The distribution function of $\sqrt{Z_n+\varepsilon}$ is given by

$$\Pr\left(\sqrt{Z_n+\varepsilon}\leq x\right)=\Pr(Z_n\leq x^2-\varepsilon)=1-\exp(-(x^2-\varepsilon)/\mu_n)$$

for $x\geq\sqrt{\varepsilon}$. Hence $E(\sqrt{Z_n+\varepsilon})<\infty$ and

$$E(D_n)=\frac{(p-1/n)}{2}E\left(\sqrt{Z_n+\varepsilon}\right)-\frac{(p-1/n)}{2}E\left(\sqrt{Z_n+\varepsilon}\right)=0$$

for all $n>\varepsilon/k$. Moreover,

$$E(D_n^2)=(p-1/n)E(Z_n+\varepsilon)=(p-1/n)\left(\frac{\varepsilon/n}{p-1/n}+\varepsilon\right)=p\,\varepsilon=k$$

for all $n>\varepsilon/k$.