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Geometric ergodicity of the Gibbs sampler for the Poisson change-point model

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ABSTRACT

Poisson change-point models have been widely used for modelling inhomogeneous time-series of count data. There are a number of methods available for estimating the parameters in these models using iterative techniques such as MCMC. Many of these techniques share the common problem that there does not seem to be a definitive way of knowing the number of iterations required to obtain sufficient convergence. In this paper, we show that the Gibbs sampler of the Poisson change-point model is geometrically ergodic. Establishing geometric ergodicity is crucial from a practical point of view as it implies the existence of a Markov chain central limit theorem, which can be used to obtain standard error estimates. We prove that the transition kernel is a trace-class operator, which implies geometric ergodicity of the sampler. We then provide a useful application of the sampler to a model for the quarterly driver fatality counts for the state of Victoria, Australia.

1. Introduction

Under the Poisson change-point model we observe a nonhomogeneous sequence of T independent Poisson random variables $X_1, \ldots, X_T$. More specifically, we consider the case when the rate, λ, changes from λ_1 to λ_2 at an unknown point τ_1, then from λ_2 to λ_3 at a later unknown point τ_2, and so on, until the rate changes to λ_K, where it remains, for the observation periods $\tau_K + 1$ to T. This model has been widely studied (see Carlin et al., 1992, and Raftery and Akman, 1986, among others). Here, we use a Poisson change-point model for detecting the shifts and levels of quarterly driver fatality counts for the state of Victoria, Australia. Within this application, the timing and size of the shifts in the dynamics of the data provide insight into the effectiveness of particular government policies in reducing the number of road fatalities. In this paper, we utilize the results from Khare and Hobert (2011) to show a theoretical result on the convergence of the Gibbs sampler for estimating the model parameters that is of great importance to practitioners. In cases where these models are utilized for providing objective evidence to influence future policy-making, we must have confidence that the iterative algorithm for estimating the model parameters has converged.

In Section 2, we outline the model specification and introduce some notations. We then discuss the estimation of the model parameters in Section 3. The main result is presented in Section 4 where we show that the Gibbs sampler is geometrically ergodic. These theoretical results are used in practice in Section 5, where we apply the model to the quarterly driver fatality counts for the state of Victoria, Australia. We then discuss some conclusions and potential avenues for future research.

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2. Model specification

Consider the Poisson change-point model, where

$$Y_i|\boldsymbol{\lambda}, \tau \overset{\text{ind.}}{\sim} \begin{cases} \text{Po}(\lambda_1) & \text{for } i = 1, \dots, \tau_1; \\ \text{Po}(\lambda_2) & \text{for } i = \tau_1 + 1, \dots, \tau_2; \\ \vdots & \\ \text{Po}(\lambda_K) & \text{for } i = \tau_{K-1} + 1, \dots, \tau_K; \\ \text{Po}(\lambda_{K+1}) & \text{for } i = \tau_K + 1, \dots, T. \end{cases} \tag{1}$$

$$\lambda_i|\boldsymbol{\beta}, \tau \overset{\text{ind.}}{\sim} \text{G}(a_i, \beta_i), \quad i = 1, \dots, K+1$$

$$\beta_i|\tau \overset{\text{ind.}}{\sim} \text{IG}(c_i, d_i), \quad i = 1, \dots, K+1$$

$$\tau_1, \dots, \tau_K \quad \text{are sampled without replacement from the set } \{1, 2, \dots, T-1\}.$$

Here $X \sim \text{G}(a, b)$ implies that we have the density $f_X(x) = \frac{x^{a-1}}{b^a \Gamma(a)} e^{-\frac{x}{b}},\ x > 0$ and $X \sim \text{IG}(c, d)$ implies that we have the density $f_X(x) = \frac{1}{d^c x^{c+1} \Gamma(c)} e^{-\frac{1}{dx}},\ x > 0$. Without loss of generality, we assume $\tau_1 < \tau_2 < \cdots < \tau_K$. Note also that $1 \le \tau_1 \le T-K-1$ and $\tau_{i-1}+1 \le \tau_i \le T-K-1+i$ for $i = 2, \dots, K$, which implies there must be K change-points in the observation period. If we fix $K = 1$, then we have the Poisson change-point model that was studied in Carlin et al. (1992).

3. Estimation of the model parameters

Our main interest is in estimating the vector $\boldsymbol{\lambda}$ and the change-points $\boldsymbol{\tau} = (\tau_1, \dots, \tau_K)$ by obtaining a sample from their posterior distributions. From (1) we obtain the joint density

$$f(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\beta}, \boldsymbol{\tau}) = \frac{1}{\binom{T-1}{K}} \prod_{h=1}^{\tau_1} \frac{\lambda_1^{y_h} e^{-\lambda_1}}{y_h!} \prod_{i=2}^{K} \left\{ \prod_{j=\tau_{i-1}+1}^{\tau_i} \frac{\lambda_i^{y_j} e^{-\lambda_i}}{y_j!} \right\} \prod_{k=\tau_K+1}^{T} \frac{\lambda_K^{y_k} e^{-\lambda_K}}{y_k!} \prod_{l=1}^{K+1} \frac{\lambda_l^{a_l-1}}{\beta_l^{a_l} \Gamma(a_l)} e^{-\frac{\lambda_l}{\beta_l}}$$
$$\times \prod_{m=1}^{K+1} \frac{1}{d_m^{c_m} \beta_m^{c_m+1} \Gamma(c_m)} e^{-\frac{1}{d_m \beta_m}}. \tag{2}$$

Then, the complete posterior density is

$$f(\boldsymbol{\lambda}, \boldsymbol{\beta}, \boldsymbol{\tau}|\boldsymbol{y}) \propto \lambda_1^{\sum_{i=1}^{\tau_1} y_i + a_1 - 1} \prod_{k=2}^{K} \left\{ \lambda_{k+1}^{\sum_{i=\tau_{k-1}+1}^{\tau_k} y_i + a_{k+1} - 1} \right\} \lambda_{K+1}^{\sum_{i=\tau_K+1}^{T} y_i + a_2 - 1}$$
$$\times \frac{e^{-\lambda_1\left(\tau_1 + \frac{1}{\beta_1}\right)} \prod_{k=2}^{K} \left\{ e^{-\lambda_k(\tau_k - \tau_{k-1} + \frac{1}{\beta_k})} \right\} e^{-\lambda_{K+1}\left(T - \tau_{K-1} + \frac{1}{\beta_{K+1}}\right)} \prod_{k=1}^{K+1} \left\{ e^{-\frac{1}{d_k \beta_k}} \right\}}{\beta_1^{a_1+c_1+1} \beta_2^{a_2+c_2+1}}.$$

The desired sample will be obtained by running a two-stage Gibbs sampler that iterates between

$$f(\boldsymbol{\lambda}|\boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{y}) \quad \text{and} \quad f(\boldsymbol{\beta}, \boldsymbol{\tau}|\boldsymbol{\lambda}, \boldsymbol{y}),$$

where the sequence of $\boldsymbol{\beta}$'s will be simply ignored.

From (2), it is clear that conditional on $\boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{y}$, the parameters $\lambda_1, \dots, \lambda_{K+1}$ are independent with

$$\begin{aligned} \lambda_1|\boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{y} &\sim \text{G}\left(\sum_{i=1}^{\tau_1} y_i + a_1, \frac{\beta_1}{\tau_1 \beta_1 + 1} \right); \\ \lambda_k|\boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{y} &\sim \text{G}\left(\sum_{i=\tau_{k-1}+1}^{\tau_k} y_i + a_k, \frac{\beta_k}{(\tau_k - \tau_{k-1})\beta_k + 1} \right) \quad \text{for } k = 2, \dots, K; \\ \lambda_{K+1}|\boldsymbol{\beta}, \tau, \boldsymbol{y} &\sim \text{G}\left(\sum_{i=\tau_K+1}^{T} y_i + a_{K+1}, \frac{\beta_{K+1}}{(T - \tau_K)\beta_{K+1} + 1} \right). \end{aligned} \tag{3}$$

Again, from (2) it is clear that conditional on $\boldsymbol{\lambda}, \boldsymbol{y}$, the parameters $\beta_1, \dots, \beta_{K+1}$ and $\boldsymbol{\tau}$ are independent with

$$\begin{aligned} \beta_k|\boldsymbol{\lambda}, \boldsymbol{y} &\sim \text{IG}\left(a_k + c_k, \frac{d_k}{d_k \lambda_k + 1} \right) \quad \text{for } k = 1, \dots, K+1; \\ f(\boldsymbol{\tau}|\boldsymbol{\lambda}, \boldsymbol{y}) &= \frac{f(\boldsymbol{y}|\boldsymbol{\tau}, \boldsymbol{\lambda})}{\sum_{\tau_1'=1}^{T-K-1} \sum_{i=2}^{K} \sum_{\tau_i'=\tau_{i-1}'+1}^{T-K-1+i} f(\boldsymbol{y}|\boldsymbol{\tau}', \boldsymbol{\lambda})}. \end{aligned} \tag{4}$$