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Parameter estimation in two-type continuous-state branching processes with immigration

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ABSTRACT

We study the parameter estimation of two-type continuous-state branching processes with immigration based on low frequency observations at equidistant time points. The ergodicity of the processes is proved. The estimators are based on the minimization of a sum of squared deviation about conditional expectations. We also establish the strong consistency and central limit theorems of the conditional least squares estimators and the weighted conditional least squares estimators of the drift and diffusion coefficients based on low frequency observations.

1. Introduction

Branching processes have been used widely not only in biology, but also in financial world. For example, Galton–Watson branching processes with immigration (GWI-processes) are used to study the evolution of different species. Continuous-state branching processes (CB-processes) were first introduced by Jiřina (1958). In particular, a continuous CB-process can be obtained as the unique solution of a stochastic equation system driven by Brownian motion. Kawazu and Watanabe (1971) constructed continuous-state branching processes with immigration (CBI-processes). In view of the results of Dawson and Li (2006), a general single-type CBI-process is the unique strong solution of a stochastic equation driven by Brownian motions and Poisson random measures. The two-type CB-processes were first introduced by Watanabe (1969). Ma (2012) proved the existence and uniqueness of the strong solution of a two-dimensional stochastic integral equation system with jumps. He also showed that the unique solution is a two-type CBI-process. In financial world, multitype CBI-processes are used to describe the relations of the prices of different assets and interest rates of different currencies.

First, we introduce a special continuous single-type CBI-process defined by the following equation:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t, \tag{1.1}$$

where $(a, b, \sigma) \in (0, +\infty)^3$ and B_t is a standard Brownian motion. In fact, the solution of (1.1) is also known as the *Cox–Ingersoll–Ross model* (CIR-model) introduced by Cox et al. (1985) for the term structure of interest rates. The above equation was also studied in Ikeda and Watanabe (1989) and Revuz and Yor (1991). The basic theory of general CBI-processes was

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developed in Li (2011). The appealing properties of this process are as follows:

(1) The process stays nonnegative.
(2) It converges to a steady-state law with mean a/b, the so-called long-term value, with speed of adjustment b.
(3) The incremental variance is proportional to its current value.

However, the one-dimensional CIR-model does not describe the connection among interest rates of different currencies. In order to give more objective description to the financial environment, we need to deal with the multifactor CIR-model or the multitype CBI-processes. In order to make the presentation easier, we just analyze the two-type CBI-processes defined by the following equation:

$$\begin{cases} dX_1(t) = (a_1 - b_{11}X_1(t) + b_{12}X_2(t))dt + \sigma_1\sqrt{X_1(t)}dB_1(t), \\ dX_2(t) = (a_2 + b_{21}X_1(t) - b_{22}X_2(t))dt + \sigma_2\sqrt{X_2(t)}dB_2(t), \end{cases} \tag{1.2}$$

where $\theta = (a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22}, \sigma_1, \sigma_2) \in \mathbf{S}$ and $\mathbf{S} = (0, +\infty)^3 \times [0, \infty)^2 \times (0, \infty)^3$. $B_i(t)$, $i = 1, 2$ are independent standard Brownian motions. The existence and uniqueness of the solution to (1.2) were proved in Ma (2012). We can rewrite (1.2) into the vector form:

$$dX_t = (A - BX_t)dt + \Sigma\sqrt{X_t}dW_t, \tag{1.3}$$

where

$$X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}, \qquad A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{pmatrix},$$
$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad \sqrt{X_t} = \begin{pmatrix} \sqrt{X_1(t)} & 0 \\ 0 & \sqrt{X_2(t)} \end{pmatrix}, \qquad W_t = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}.$$

We can use a special form of (1.2) to describe the relations among the interest rates of different currencies. In currency market, we assume $X_1(t)$ is the interest rate of a very strong and influential currency, and $X_2(t)$ is the interest rate of a less influential currency. The situation can be described by the following stochastic equation:

$$\begin{cases} dX_1(t) = b_{11}\left(\dfrac{a_1}{b_{11}} - X_1(t)\right)dt + \sigma_1\sqrt{X_1(t)}dB_1(t), \\ dX_2(t) = b_{22}\left(\dfrac{a_2}{b_{22}} + \dfrac{b_{21}}{b_{22}}X_1(t) - X_2(t)\right)dt + \sigma_2\sqrt{X_2(t)}dB_2(t). \end{cases}$$

Here, the first equation gives the evolution of $X_1(t)$, which is just a one-dimensional CIR-model. The second equation describes the evolution of $X_2(t)$, which is affected not only by the random noise, but also by $X_1(t)$. Specifically, the second coordination $X_2(t)$ has the following properties:

(1) It stays nonnegative.
(2) It converges to a steady-state law with mean $a_2/b_{22} + (b_{21}/b_{22})(a_1/b_{11})$ (this can be easily got from (3.1) with $t \to \infty$), the so-called long-term value, where the second term is contributed by $X_1(t)$.
(3) Its incremental variance is proportional to its current value.

If $b_{12}b_{21} \neq 0$, then (1.2) can account for the fluctuations of the rate of two currencies that affect with each other. Furthermore, CBI processes are the scaling limits of discrete Galton–Watson processes with immigration, see Lamperti (1967) and Li (2011). (1.2) can also be used to represent genetic types in an animal population, and mutant types in a bacterial population. For work in this direction, see, for example Jiřina (1958) and Li (2011). Roughly speaking,

(1) b_{11}, b_{12} can be interpreted to be the death rate of the first type and the transformation speed from the second type to the first type, respectively. b_{21}, b_{22} can be interpreted similarly.
(2) The two type will stay nonnegative and converge to a steady-state law with mean $B^{-1}A$.

However, before using (1.3) to solve the practical problems, we need to estimate the parameters in the equation based on the historical information. For the single-type CBI-processes, the approaches to parameter estimation can be found in Longstaff and Schwartz (1992) and Bibby and Sørensen (1995). Overbeck and Rydén (1997) also gave the *conditional least squares estimators* (CLSEs). Estimators of the matrix of offspring means and the vector of stationary means in a multitype GWI-process had been given in Quine and Durham (1977). For multitype GWI-process, the *weighted conditional least squares estimator* (WCLSE) of the mean matrix was developed in Shete and Sriram (2003). The asymptotical properties of CLSEs of GWI-processes with general offspring laws were studied in Venkataraman (1982) and Wei and Winnicki (1989). The asymptotics of CLSEs and WCLSEs of a stable CIR-model was studied in Li and Ma (2013). It is well-known that the CBI-processes are special examples of the affine Markov processes studied in Duffie et al. (2003). The ergodicity and estimation of some different two-dimensional affine processes were studied in Barczy et al. (2013, in press-a,b).

In this work, we give the CLSEs and the WCLSEs of the parameters in (1.3) using low frequency observations at equidistant time points $\{k\Delta : k = 0, 1, \ldots, n\}$ of a single realization $\{X_t : t \geq 0\}$, where $X_t = (X_1(t), X_2(t))^{\mathrm{T}}$. For simplicity, we take