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Closure property and maximum of randomly weighted sums with heavy-tailed increments[☆]

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ABSTRACT

In this paper, we consider the randomly weighted sum $S_2^\Theta = \Theta_1 X_1 + \Theta_2 X_2$, where the two primary random summands X_1 and X_2 are real-valued and dependent with long or dominatedly varying tails, and the random weights Θ_1 and Θ_2 are positive, with values in $[a, b]$, $0 < a \le b < \infty$, and arbitrarily dependent, but independent of X_1 and X_2. Under some dependence structure between X_1 and X_2, we show that S_2^Θ has a long or dominatedly varying tail as well, and obtain the corresponding (weak) equivalence results between the tails of S_2^Θ and $M_2^\Theta = \max\{\Theta_1 X_1, \Theta_1 X_1 + \Theta_2 X_2\}$. As corollaries, we establish the asymptotic (weak) equivalence formulas for the tail probabilities of randomly weighted sums of even number of long-tailed or dominatedly varying-tailed random variables.

1. Introduction

Throughout this paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $u(x)$ and $v(x)$, we write $u(x) \sim v(x)$ if $\lim u(x)/v(x) = 1$; write $u(x) \lesssim v(x)$ if $\limsup u(x)/v(x) \le 1$; write $u(x) \gtrsim v(x)$ if $\liminf u(x)/v(x) \ge 1$. For a real number x, write $x^+ = \max\{x, 0\}$. The indicator function of an event A is denoted by $\mathbf{1}_A$. For any distribution V, we assume that its tail distribution $\overline{V}(x) = 1 - V(x) > 0$ for all x. The Lebesgue–Stieltjes integrals $\int_{(u,v]}$ and $\int_{[u,v]}$, where $-\infty \le u < v < \infty$, are denoted by $\int_u^v$, $\int_{u-}^v$, respectively.

A distribution V is called long-tailed, denoted by $V \in \mathscr{L}$, if $\overline{V}(x+y) \sim \overline{V}(x)$ holds for every fixed y, and is called dominatedly varying-tailed, denoted by $V \in \mathscr{D}$, if $\limsup_{x\to\infty} \overline{V}(xy)/\overline{V}(x) < \infty$ for any $y \in (0, 1)$.

Let X_1 and X_2 be two dependent real-valued random variables (r.v.s) with distributions F_1 and F_2, respectively; let Θ_1, Θ_2 be two arbitrarily dependent r.v.s., independent of X_1 and X_2, and there exist some constants $0 < a \le b < \infty$ such that

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$\mathrm{P}(a \le \Theta_k \le b) = 1$, $k = 1, 2$. Denote the randomly weighted sum and its maximum, respectively, by

$$S_2^{\Theta} := \Theta_1 X_1 + \Theta_2 X_2 \quad \text{and} \quad M_2^{\Theta} := \max\{\Theta_1 X_1, \Theta_1 X_1 + \Theta_2 X_2\}. \tag{1.1}$$

In this paper, we focus on two main questions. Firstly, we are interested in the closure property of the sum S_2^{Θ}, which states that, given F_1 and F_2 from some heavy-tailed distribution class, the distribution function (d.f.) of the sum S_2^{Θ} belongs to the same class. Secondly, we are interested in the (weak) equivalence result for the tail probability of M_2^{Θ}. In Theorem 2.1 below we prove the aforementioned closure property and corresponding equivalence result

$$\mathrm{P}(M_2^{\Theta} > x) \sim \mathrm{P}(S_2^{\Theta} > x) \tag{1.2}$$

in the case of long-tailed distributions and dependence structure between X_1 and X_2 given by Assumption A below. In Theorem 2.2, we prove analogous result in the case of dominatedly varying-tailed distributions.

Historically, the above questions have been considered mostly for $\Theta_1 = \Theta_2 = 1$ and independent variables case, see Embrechts and Goldie (1980), Leslie (1989), Sgibnev (1996), Ng et al. (2002), Tang and Tsitsiashvili (2003a), Cai and Tang (2004), Geluk and Ng (2006) and Foss et al. (2009) among others. In particular, Embrechts and Goldie (1980) proved the convolution closure of the long-tailed distributions, whereas the closure of the dominatedly varying-tailed distributions was proved in Cai and Tang (2004) and Watanabe and Yamamuro (2010). In fact, in the case when $F_k \in \mathscr{D}$, $k = 1, 2$, are supported on $[0, \infty)$, the closure property is valid for any (not necessarily independent) r.v.s X_1 and X_2 (see the proof of Proposition 2.1 in Cai and Tang (2004)). Note that, for independent variables, the closure property is linked to the so-called max-sum equivalence relation $\mathrm{P}(X_1 + X_2 > x) \sim \mathrm{P}(\max\{X_1, X_2\} > x)$. In the case of subexponential distributions on $[0, \infty)$, Embrechts and Goldie (1980) showed that these two properties are equivalent. Sgibnev (1996) related the tail $\mathrm{P}(M_2^{\Theta} > x)$ to the probability $\mathrm{P}(X_1 > x)$ in the case where $F_1 \equiv F_2$ belongs to the generalized subexponential class $\mathscr{S}(\gamma)$, Ng et al. (2002) gave a generalization to the subexponential class $\mathscr{S} = \mathscr{S}(0)$, whereas Geluk and Ng (2006) extended this relation to the class of long-tailed distributions.

The mentioned above closure property and tail-equivalence (1.2) can be extended by induction to the case of $n > 2$ randomly weighted variables $\Theta_1 X_1, \ldots, \Theta_n X_n$, when $X_1, \ldots, X_n$ are *independent* r.v.s and $\Theta_1, \ldots, \Theta_n$ are nonnegative bounded r.v.s., independent of X_k's, see Tang and Tsitsiashvili (2003b), Chen and Yuen (2009), Gao and Wang (2010), Tang et al. (2011), Chen et al. (2011), Yang et al. (2012), etc. The motivation for these studies come mainly from the insurance risk theory, studying the so-called discrete-time stochastic risk model with insurance risk and financial risks, introduced by Tang and Tsitsiashvili (2003a). In such a model, each X_k is interpreted as the net loss (the total claim amount minus the total premium income) of an insurance company during period k, Θ_k is the corresponding stochastic discount factor to the origin, $S_n^{\Theta} := \sum_{k=1}^{n} \Theta_k X_k$ and $M_n^{\Theta} := \max_{1 \le k \le n} S_k^{\Theta}$ represent the stochastic present value of the aggregate net losses and the maximal net loss during the first n periods.

Recently, Chen et al. (2011) proved that in the case where the independent but not necessarily identically distributed r.v.s $X_1, \ldots, X_n$ are long-tailed, and there exist some constants $0 < a \le b < \infty$ such that $\mathrm{P}(a \le \Theta_k \le b) = 1$ for each $1 \le k \le n$, it holds that

$$\mathrm{P}(M_n^{\Theta} > x) \sim \mathrm{P}(S_n^{\Theta} > x) \sim \mathrm{P}(S_n^{\Theta+} > x), \tag{1.3}$$

where $S_n^{\Theta+} := \sum_{k=1}^{n} \Theta_k X_k^{+}$. Relations (1.3) are not only of theoretical interest but also have some practical implications. For instance, if we need to calculate the distribution tail of maximum M_n^{Θ} for large x, we can reduce the calculation to that of sum S_n^{Θ} or $S_n^{\Theta+}$.

In this paper, we consider the case when the r.v.s X_1 and X_2 in (1.1) are such that the d.f.s $F_k \in \mathscr{L}$ or $F_k \in \mathscr{D}$, $k = 1, 2$, together with the following dependence structure between X_1 and X_2:

$$\begin{aligned} \mathrm{P}(X_2 > x | X_1 = y) \;&\sim\; h_1(y)\overline{F_2}(x), \\ \mathrm{P}(X_1 > x | X_2 = y) \;&\sim\; h_2(y)\overline{F_1}(x), \end{aligned} \tag{1.4}$$

uniformly for all $y \in \mathbb{R}$, where $h_k(\cdot) : \mathbb{R} \mapsto \mathbb{R}_+ := (0, \infty)$, $k = 1, 2$ are measurable functions and the uniformity is understood as

$$\lim_{x \to \infty} \sup_{y \in \mathbb{R}} \left| \frac{\mathrm{P}(X_i > x | X_j = y)}{h_j(y)\overline{F_i}(x)} - 1 \right| = 0, \quad i, j = 1, 2,\ i \ne j.$$

When y is not a possible value of X_j, the conditional probability in (1.4) is understood as unconditional and therefore $h_j(y) = 1$ for such y. Clearly, the uniformity in (1.4) implies $\mathrm{E}h_1(X_1) = \mathrm{E}h_2(X_2) = 1$. If X_1 and X_2 are independent, then $h_1(y) = h_2(y) \equiv 1$. Dependence structure in (1.4) was proposed by Asimit and Badescu (2010). Some examples of distributions $\mathrm{P}(X_1 \le x_1, X_2 \le x_2) = C(F_1(x_1), F_2(x_2))$ (with continuous F_1 and F_2) and corresponding functions $h_1(\cdot)$, $h_2(\cdot)$ satisfying (1.4) can be found in Asimit and Badescu (2010) and Li et al. (2010). Relations (1.4) are easy to verify for some well-known bivariate copulas, allowing both positive and negative dependence, and are convenient tools when dealing with the tail behavior of the sum of two dependent r.v.s. In the case of Ali–Mikhail–Haq copula of the form

$$C(u, v) = \frac{uv}{1 - r(1-u)(1-v)}, \quad r \in (-1, 1),$$