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Estimations and asymptotic behaviors of coherent entropic risk measure for sums of random variables

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A B S T R A C T

In this article, we provide an estimation and several asymptotic behaviors for the coherent entropic risk measure of compound Poisson process. We also establish an estimation for the coherent entropic risk measure of sum of i.i.d. random variables in virtue of Log-Sobolev inequality. As an application, we provide two deviation estimations of the tail probability for compound Poisson process. Finally, several simulation results are given to support our results.

1. Introduction

Quantifying the risk of uncertainty is an important task for risk management. Researchers introduce the concept of risk measure to quantify the risk of uncertainty, and a risk measure is defined as a mapping from a set of random variables to the set of real numbers. There are many kinds of risk measures such as value at risk, and average value at risk which is also called conditional value at risk, see Föllmer and Schied (2004).

In this paper, we consider a risk measure called entropic risk measure (Föllmer and Knispel, 2011) defined as follows. Consider a random variable X defined on probability space $(\Omega, \mathcal{F}, P)$ with distribution μ, and valued in a linear space $\mathbb{X}$ of bounded measurable functions. The entropic risk measure of X with parameter $\theta > 0$ is defined by

$$ERM_\theta(X) := \frac{1}{\theta} \log E\left(e^{-\theta X}\right), \tag{1.1}$$

where $E(e^{-\theta X})$ and $\log E(e^{-\theta X})$ are usually called moment generating function and cumulant generating function respectively (see Dembo and Zeitouni (1998) for example). It is known that $ERM_\theta(X)$ has the following robust representation (Ahmadi-Javid, 2012; Föllmer and Knispel, 2011):

$$ERM_\theta(X) = \sup_{Q \in \mathcal{M}} \left\{ E_Q(-X) - \frac{1}{\theta} I_{\mathcal{F}}(Q, P) \right\}, \tag{1.2}$$

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where $\mathcal{M}$ denotes the collection of all the probability measures on $\mathbb{X}$, and $E_Q(\cdot)$ means the mathematical expectation with respect to Q. For simplicity, we use $E(\cdot)$ instead of $E_P(\cdot)$ to represent the mathematical expectation under P. The notation $I_{\mathcal{F}}(Q, P)$ stands for the relative entropy with

$$I_{\mathcal{F}}(Q, P) = \begin{cases} E\left(\frac{dQ}{dP}\log\frac{dQ}{dP}\right), & Q \ll P, \qquad \text{(a)} \\ +\infty, & \text{otherwise} \qquad \text{(b)}. \end{cases} \tag{1.3}$$

By the robust representation of $ERM_\theta(\cdot)$, we know $ERM_\theta(\cdot)$ is a convex risk measure since $ERM_\theta(\cdot)$ is the dual of $I_{\mathcal{F}}(\cdot, P)/\theta$ and $I_{\mathcal{F}}(\cdot, P)/\theta$ is convex.

The convex entropic risk measure $ERM_\theta(\cdot)$ is related to the ranking functions (see Di Pierro and Mosevich (2011)) through the exponential utility function $u(x) := -e^{-\theta x}/\theta$. The ranking functions are usually used in portfolio selections. Di Pierro and Mosevich (2011) consider the ranking function defined by $R_u(X) := E(u(X))$. They indicate that the ranking function defined by $R_*(X) := -\log(-R_u(X))/\theta$ is equivalent to $R_u(X)$, and

$$R_*(X) = -ERM_\theta(X). \tag{1.4}$$

For recent contribution, we refer to Hürlimann (2013a,b,c). The risk measure $ERM_\theta(\cdot)$ is also related to the premium calculation principles (Deprez and Gerber, 1985) and the large deviation principles (Dembo and Zeitouni, 1998) through the exponential utility function.

The risk measure $ERM_\theta(\cdot)$ is convex, but it is not coherent because $ERM_\theta(\cdot)$ is not positively homogeneous. Hence, Föllmer and Knispel consider the coherent entropic risk measure (Föllmer and Knispel, 2011) with level $\alpha > 0$, which is defined by

$$CERM_\alpha(X) := \sup_{Q\in\mathcal{M}, I_{\mathcal{F}}(Q,P)\le\alpha} E_Q(-X). \tag{1.5}$$

It is known that $CERM_\alpha$ is coherent (cf. Page 337 of Föllmer and Knispel, 2011) and

$$CERM_\alpha(X) = \inf_{\theta>0}\left\{\frac{\alpha}{\theta} + ERM_\theta(X)\right\}. \tag{1.6}$$

In quantitative risk management, we usually need to quantify the risks of $\sum_{k=1}^{n} X_k$ and $\sum_{k=1}^{N(t)} X_k$, where X_k $(k \ge 1)$ are the independent copies of random variable X. The process $\{N(t), t \ge 0\}$ is a Poisson process with intensity λ. We assume that $\{N(t), t \ge 0\}$ and $\{X_k, k \ge 1\}$ are independent. Generally, $M(t) := \sum_{k=1}^{N(t)} X_k$ is called compound Poisson process. Föllmer and Knispel (2011) provide the estimations and asymptotic behaviors of several risk measures for random variables and sum of i.i.d. variables. They obtain, for $\gamma \in (0, 1)$

$$V@R_\gamma(X) \le V@R_{\gamma-}(X) \le AV@R_\gamma(X) \le CERM_{-\log(\gamma)}(X), \tag{1.7}$$

where $V@R$ is the value at risk, $AV@R$ is the average value at risk. They also obtain

$$\lim_{n\to+\infty} \frac{1}{n} CERM_\alpha\left(\sum_{i=1}^{n} X_i\right) = -E(X), \tag{1.8}$$

and

$$\lim_{n\to\infty} \sqrt{n}\left(\frac{1}{n} CERM_\alpha\left(\sum_{i=1}^{n} X_i\right) + E(X)\right) = \sqrt{2\alpha Var(X)}. \tag{1.9}$$

Inspired by above (1.7)–(1.9), we provide a bound and several asymptotic behaviors for $CERM_\alpha(M(t))/t$. As an application of the bound, we obtain two deviation estimations for the tail probabilities of $M(t)$. For the sum of i.i.d. random variables $\sum_{i=1}^{n} X_i$, we also establish a bound for $CERM_\alpha(\sum_{i=1}^{n} X_i)/n$ with respect to n by virtue of Log-Sobolev inequality.

2. Estimation and asymptotic behaviors of coherent entropic risk measure for compound Poisson process and their applications

2.1. Estimation and asymptotic behaviors of coherent entropic risk measure for compound Poisson process

In this subsection, we establish an estimation and several asymptotic behaviors of $CERM_\alpha(M(t))/t$. Precisely, in Theorem 2.1 we provide an estimation for $CERM_\alpha(M(t))/t$. In Corollary 2.1 and Theorem 2.2, we provide two asymptotic behaviors for $CERM_\alpha(M(t))/t$ as t goes into infinity. In comparison to the classical limit theories, Theorem 2.1, Corollary 2.1 and Theorem 2.2 are similar to the Berry Esseen estimation, law of large numbers and central limit theorem respectively.

Theorem 2.1. *Assume for any $\delta > 0$, $E(e^{\delta|X|}) < +\infty$. The following bound holds*

$$\left|\frac{1}{t} CERM_\alpha(M(t)) + \lambda E(X)\right| \le \frac{3\sqrt{\lambda\alpha E(X^2)}}{\sqrt{t}}. \tag{2.1}$$