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On the behavior of the log Laplace transform of series of weighted non-negative random variables at infinity☆

Leonid Rozovsky

St. Petersburg Chemical Pharmaceutical Academy, Department of Mathematics, Prof. Popov St. 14, 197376, St. Petersburg, Russia

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ABSTRACT

We obtain the new asymptotics of the log Laplace transform of $\sum_{j\geq 1} \lambda_j X_j$ at infinity, where $\{X_j\}$ are i.i.d. non-negative random variables and $\{\lambda_j\}$ is a sequence of positive and non-increasing numbers, satisfying certain regularity conditions.

1. Introduction

Let $S = \sum_{j\geq 1} \lambda_j X_j$, where $\{\lambda_j\}$ is a sequence of *non-increasing* positive numbers and $\{X_i\}$ are independent copies of a non-negative random variable X with a distribution function $V(x)$.

We assume that $\mathbf{P}(S < \infty) = 1$ (or, equivalently, $\sum_{j\geq 1} \mathbf{E} \min(1, \lambda_j X) < \infty$) and examine the behavior of $-\log \mathbf{P}(S < r)$ as $r \to 0$.

This problem was studied by a number of authors. First of all, we refer the reader to the recent notes by Aurzada (2007, 2008), and Borovkov and Ruzankin (2008a,b) (more extensive lists of references can be found therein and also in Rozovsky, 2009a).

Let us present two main conclusions of the aforementioned authors, in which *explicit* forms of the asymptotics of $-\log \mathbf{P}(S < r)$ as $r \to 0$ were derived provided that the sequence $\{\lambda_j\}$ satisfies certain additional conditions.

Set

$$\phi(t) = \mathbf{E}e^{-tX}, \qquad \phi_S(t) = \mathbf{E}e^{-tS}, \quad t \geq 0. \tag{1.1}$$

Theorem 1 (*Borovkov and Ruzankin, 2008a, Theorem 2.1*). *Let a positive and non-increasing function $\lambda(u)$, $u > 0$, be regularly varying at ∞ with an exponent $-1/\gamma < -1$. We assume also that $\lambda_j \sim \lambda(j)$, $j \to \infty$. If $-\log \phi(t) = O(t^{\gamma-\delta})$, $t \to \infty$, and $\mathbf{E}X^{\gamma+\delta} < \infty$ for some $\delta > 0$, then $\mathbf{P}(S < \infty) = 1$ and*

$$-\log \phi_S(t) \sim K(\gamma)\, \lambda^{-1}(1/t) \quad \text{as } t \to \infty, \tag{1.2}$$

where $K(\gamma) = -\int_0^\infty u^{-\gamma} \mathrm{d}\log \phi(u) < \infty$ and $\lambda^{-1}(\cdot)$ is an inverse function of $\lambda(\cdot)$.

Theorem 1 is a generalization of an earlier result by Aurzada (2007), where relation (1.2) was obtained in the case $\lambda_j \sim j^{-1/\gamma} \log^\delta j$ under an optimal moment assumption (also see the discussion below Theorem 2.1 from Borovkov and Ruzankin, 2008a).

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E-mail address: L_Rozovsky@mail.ru.

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Theorem 2 (Aurzada, 2007). *Let $\lambda_j = q^{j-1}$ with some $0 < q < 1$ and $\mathbf{E}\log(1+X) < \infty$. If $-\log\phi(t) \sim K\,t^{\tau}$ as $t \to \infty$ for some positive K and $\tau \in (0, 1)$, then $\mathbf{P}(S < \infty) = 1$ and*

$$\log\phi_S(t) \sim (1-q^{\tau})^{-1}\log\phi(t) \quad \text{as } t \to \infty. \tag{1.3}$$

Note that we formulate the results in the terms of the Laplace transforms of X and S. It is often more convenient and looks simpler and shorter. The equivalent recalculation in the terms of distribution functions is no problem (see Rozovsky, 2009b).

We can observe that the conclusion of Theorem 1 is rather general but it includes too restrictive assumptions on V at zero and at infinity; moreover, Borovkov and Ruzankin (2008a) do not examine the situation $K(\gamma) = \infty$. In turn, (1.3) is valid under the optimal moment condition but the choice of λ_j is too specific. In other words, these results (and the most of the other results, known to the author) are incomplete to some extent.

The main objective of our paper is to present the general forms of the asymptotics of $-\log\phi_S(t)$, $t \to \infty$, which hold true under the best possible (in a certain sense, necessary and sufficient) conditions, provided that the numbers $\{\lambda_j\}$ satisfy some rather mild assumptions. The obtained asymptotics make it possible to reduce the problem under investigation to another, rather technical, problem, namely, to an asymptotic analysis of respective integrals. For example, we deduce that if $\mathbf{P}(S < \infty) = 1$ and (see the notation in Theorem 1)

$$-\log\phi(t) = o\left(\lambda^{-1}(1/t)\right), \quad t \to \infty, \tag{1.4}$$

then

$$-\log\phi_S(t) \sim \int_0^1 \log\phi(tu)\mathrm{d}\lambda^{-1}(u) \quad \text{as } t \to \infty. \tag{1.5}$$

Taking into consideration that under the assumptions of Theorem 1 the integral in (1.5) is equivalent to $K(\gamma)\,\lambda^{-1}(1/t)$ as $t \to \infty$ (see (5.2) and (5.3) in Borovkov and Ruzankin, 2008a), relation (1.2) follows.

By similar reasonings, various refinements and generalizations of relations (1.2) and (1.3) can be found as well as a number of new asymptotics. Since all this is rather extensive, here we limit ourselves to a few special examples, provided with a sketch of the proofs.

2. Results

Let a positive *non-decreasing* function $b(u) = b_\gamma(u)$, $u \geq 0$, be regularly varying at ∞ with an exponent γ, $0 \leq \gamma \leq 1$, i.e. the function $b_0(u) = u^{-\gamma}b(u)$ is slowly varying at ∞. Following Rozovsky (2009a), we assume that $b(u)$ and $\{\lambda_j\}$ are linked by the relation

$$b_\gamma(1/\lambda_j) \sim j \quad \text{as } j \to \infty \tag{2.1}$$

(thus, $b(u)$ is a mild version of $\lambda^{-1}(1/u)$).

Examples. (1) If $\lambda_j \sim (\log^{\delta} j/j)^{1/\gamma}$ with $\gamma > 0$ and some δ, then $b(u) = b_\gamma(u) = \gamma^{\delta}\,u^{\gamma}\log^{\delta} u$, $u > 1$.
(2) If $-\log\lambda_j \sim \alpha\,j^{\delta}$ with positive α and δ, then $b(u) = b_0(u) = (\alpha^{-1}\log u)^{1/\delta}$, $u > 1$.
(3) If $\lambda^{-1}(1/u)$ is regularly varying at ∞ with an exponent γ, then $b(u) = \lambda^{-1}(1/u)$, $u > 1$.

As it was noticed by Rozovsky (2009a), condition $\mathbf{P}(S < \infty) = 1$ is equivalent to

$$\int_1^{\infty}\left(u\int_u^{\infty} b(y)\mathrm{d}y/y^2\right)\mathrm{d}V(u) < \infty, \tag{2.2}$$

which in the case $0 \leq \gamma < 1$ is the same as $\mathbf{E}\,b_\gamma(X) < \infty$.

Denote $J(t) = -\int_1^{\infty}\log\phi(t/u)\mathrm{d}b_\gamma(u)$, $t \geq 0$.

Proposition 1. *Let conditions (2.1) and (2.2) hold. Also let a function $\omega(t)$, $t \geq 1$, be regularly varying at ∞ with an exponent $\tau \in [0, 1]$.*

(1) *If*

$$-\log\phi(t) = o\left(\omega(t)\right) \quad \text{as } t \to \infty, \tag{2.3}$$

then

$$-\log\phi_S(t) = (1+o(1))J(t) + o\left(\omega(t)\right) \quad \text{as } t \to \infty.$$

(2) *If*

$$-\log\phi(t) \sim \omega(t) \quad \text{as } t \to \infty \tag{2.4}$$

and

$$k_\tau = \int_1^{\infty} u^{-\tau}\mathrm{d}b(u) < \infty, \tag{2.5}$$