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Polar sets for anisotropic Gaussian random fields

Jakob Söhl

Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

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ABSTRACT

This paper studies polar sets for anisotropic Gaussian random fields, i.e. sets which a Gaussian random field does not hit almost surely. The main assumptions are that the eigenvalues of the covariance matrix are bounded from below and that the canonical metric associated with the Gaussian random field is dominated by an anisotropic metric. We deduce an upper bound for the hitting probabilities and conclude that sets with small Hausdorff dimension are polar. Moreover, the results allow for a translation of the Gaussian random field by a random field, that is independent of the Gaussian random field and whose sample functions are of bounded Hölder norm.

1. Introduction

Anisotropic Gaussian random fields arise naturally in stochastic partial differential equations, image processing, mathematical finance and other areas. Let $X = \{X(t)|t \in I \subset \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}^d$, where I is bounded. We will call X an (N, d)-*Gaussian random field*. The distance in the *canonical metric* associated with the Gaussian random field is $\sqrt{\mathbb{E}\left[\|X(s) - X(t)\|^2\right]}$, where $\|\cdot\|$ denotes the Euclidean metric. Polar sets for Gaussian random fields are investigated in Weber (1983) under the assumptions that the components are independent copies of the same random field, that the variance is constant and that $\sqrt{\mathbb{E}\left[\|X(s) - X(t)\|^2\right]} \le c\|s-t\|^\beta$ holds with constants $c, \beta > 0$. The recent works Xiao (2009) and Biermé et al. (2009) consider the anisotropic metric

$$\rho(s, t) := \sum_{j=1}^{N} |s_j - t_j|^{H_j} \tag{1}$$

with $H \in]0, 1]^N$ and assume $\sqrt{\mathbb{E}\left[\|X(s) - X(t)\|^2\right]} \le c\rho(s, t)$. In addition they require for the variance only to be bounded from below. In this paper the assumptions on the variance and on the independent copies in the components are substituted by the milder assumption that the eigenvalues of the covariance matrix are bounded from below. The random fields in the components neither need to be identically distributed nor independent. Hence, we require weaker assumptions on the dependency structure of the components of the Gaussian random field than Weber (1983), Xiao (2009) and Biermé et al. (2009). It follows from an upper bound on the hitting probabilities of X that sets with Hausdorff dimension smaller than $d - \sum_{j=1}^{N} 1/H_j$ are polar. Our results allow for a translation of the Gaussian random field X by a random field, that is independent of X and whose sample functions are Lipschitz continuous with respect to the metric ρ.

As an application we show that an estimator in Belomestny and Reiß (2006), which calibrates an exponential Lévy model by option data, is almost surely well-defined.

E-mail address: soehl@math.hu-berlin.de.

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2. Main results

Let X be an (N, d)-Gaussian random field. Recall that we suppose the index set I to be bounded. We will assume the following two conditions.

Condition 1. There is a constant $c > 0$ such that for all $s, t \in I$ we have $\sqrt{\mathbb{E}\left[\|X(s) - X(t)\|^2\right]} \le c\rho(s, t)$.

Condition 2. There is a constant $\lambda > 0$ such that for all $t \in I$ and for all $e \in \mathbb{R}^d$ with $\|e\| = 1$ we have $\mathbb{E}[(\sum_{j=1}^d e_j X_j(t))^2] \ge \lambda$.

Condition 1 bounds the canonical metric in terms of the anisotropic metric ρ. Condition 2 bounds the eigenvalues of the covariance matrix from below. It excludes, for example, cases where X takes values only in some vector subspace.

We will use a uniform modulus of continuity, see (69) in Xiao (2009, p. 167). We restate this result in the next inequality. A weaker formulation suffices for our purpose and is proved in the Appendix. Let X be an (N, d)-Gaussian random field, that satisfies Condition 1. Then there is a version X' of X and a constant $\tilde{c} > 0$ such that almost surely the following inequality holds:

$$\limsup_{\epsilon\downarrow 0} \sup_{s,t\in I, \rho(s,t)\le\epsilon} \frac{\|X'(s) - X'(t)\|}{\epsilon\sqrt{\log(\epsilon^{-1})}} \le \tilde{c}. \tag{2}$$

We will always assume that X is a version, which satisfies (2). We define by $\mathrm{Lip}_\rho(L) := \{f : I \to \mathbb{R}^d \mid \|f(s) - f(t)\| \le L\rho(s, t)\ \forall s, t \in I\}$ the *L-Lipschitz functions* with respect to the metric ρ. In each direction j the functions in $\mathrm{Lip}_\rho(L)$ are Hölder continuous with exponent H_j. We denote by $B_\rho(t, r) := \{s \in \mathbb{R}^N \mid \rho(s, t) \le r\}$ the closed ball of radius r around t.

Lemma 1. *Let X be an (N, d)-Gaussian random field, that satisfies Conditions 1 and 2. Then for each $L \ge 0$ there is a constant $C > 0$ such that for all $t \in I$, for all $r > 0$ and for all functions $f \in \mathrm{Lip}_\rho(L)$ we have*

$$\mathbb{P}\left(\inf_{s\in B_\rho(t,r)\cap I} \|X(s) - f(s)\| \le r\right) \le Cr^d. \tag{3}$$

Proof. For all integers $n \ge 1$ we define $\epsilon_n := r\exp(-2^{n+1})$ and denote by $N_n := N_\rho(B_\rho(t, r) \cap I, \epsilon_n)$ the covering number, that is the minimum number of ρ-balls with radii ϵ_n and centers in $B_\rho(t, r) \cap I$ that are needed to cover $B_\rho(t, r) \cap I$. We have the inclusion $B_\rho(t, r) \subseteq \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$. On the other hand each set $\prod_{j=1}^N [s_j, s_j + (\epsilon_n/N)^{1/H_j}]$ can be covered by a single ball with radius ϵ_n. Hence there is a constant $c_1 > 0$ independent of n such that $N_n \le \prod_{j=1}^N ((2rN/\epsilon_n)^{(1/H_j)} + 1) \le c_1 \exp(Q2^{n+1})$ where $Q = \sum_{j=1}^N 1/H_j$.

We denote by $\{t_i^{(n)} \in B_\rho(t, r) \cap I \mid 1 \le i \le N_n\}$ a set of points such that the balls with the centers $\{t_i^{(n)}\}$ and radii ϵ_n cover $B_\rho(t, r) \cap I$. We define

$$r_n := \beta\epsilon_n 2^{\frac{n+1}{2}},$$

where $\beta > \tilde{c}$ is some constant to be determined later. For all integers $n, k \ge 1$ and $1 \le i \le N_k$, we define the following events

$$A_i^{(k)} := \left\{\|X(t_i^{(k)}) - f(t_i^{(k)})\| \le r + \sum_{l=k}^{\infty} r_l\right\}, \tag{4}$$

$$A^{(n)} := \bigcup_{k=1}^{n} \bigcup_{i=1}^{N_k} A_i^{(k)} = A^{(n-1)} \cup \bigcup_{i=1}^{N_n} A_i^{(n)}, \tag{5}$$

where the last equality only holds for $n \ge 2$. We will show that the probability in (3) can be dominated by the limit of the probabilities of the sets $A^{(n)}$

$$\mathbb{P}\left(\inf_{s\in B_\rho(t,r)\cap I} \|X(s) - f(s)\| \le r\right) \le \lim_{n\to\infty} \mathbb{P}(A^{(n)}). \tag{6}$$

For all $s \in B_\rho(t, r) \cap I$ and all $n \ge 1$ there exists i_n such that $\rho(s, t_{i_n}^{(n)}) \le \epsilon_n$. By (2) we obtain almost surely

$$\limsup_{n\to\infty} \sup_{s\in I} \frac{\|X(s) - X(t_{i_n}^{(n)})\|}{r_n} \le \frac{\tilde{c}}{\beta} < 1,$$