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Tail behaviour of β-TARCH models

Péter Elek*, László Márkus

Department of Probability Theory and Statistics, Eötvös Loránd University, H-1117 Budapest, Pázmány Péter sétány 1/c, Hungary

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ABSTRACT

It is now common knowledge that the simple quadratic ARCH process has a regularly varying tail even when generated by a normally distributed noise, and the tail behaviour is well-understood under more general conditions as well. Much less studied is the case of β-ARCH-type processes, i.e. when the conditional variance is a 2β-power function with $0 < \beta < 1$. Opting for a little more generality and allowing for asymmetry, we consider threshold β-ARCH models, driven by noises with Weibull-like tails. (Special cases include the Gaussian and the Laplace distributions.) We show that the process generated has an approximately Weibull-like tail, too, albeit with a different exponent: $1 - \beta$ times that of the noise, in the sense that the tail can be bounded from both sides by Weibull distributions of this exponent but slightly different constants. The proof is based on taking an appropriate auxiliary sequence and then applying the general result of Klüppelberg and Lindner (2005) for the tail of infinite MA sequences with light-tailed innovations.

1. Introduction

In this paper we examine the tail behaviour of the stationary distribution of certain ARCH-type models defined by the equation

$$X_t = (\omega + \alpha_+ (X_{t-1}^+)^{2\beta} + \alpha_- (X_{t-1}^-)^{2\beta})^{1/2} Z_t, \tag{1}$$

where we apply the usual notation $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$. The model parameters ω, α_+ and α_- satisfy $\min(\alpha_+, \alpha_-) \geq 0$, $\max(\alpha_+, \alpha_-) > 0$ and $\omega > 0$, and Z_t is an i.i.d. sequence with zero mean and finite variance.

An important feature of this process is that – if the autocorrelation function is defined at all – it is an uncorrelated but not an independent sequence because its conditional variance is changing over time as a function of the lagged values (conditional heteroscedasticity). If $\beta = 1$ and $\alpha_+ = \alpha_- > 0$, we obtain the well-known ARCH (autoregressive conditionally heteroscedastic) model (Engle, 1982), where the conditional variance has a quadratic functional form. Since this process can reproduce the stylised facts (e.g. uncorrelatedness, conditional heteroscedasticity, nonnormality) of financial time series in an easily estimable way, it has become a basic tool in financial econometrics in the past two decades, and gave rise to various generalisations. For instance, in order to model the fact that the variance of stock returns responds more strongly to negative shocks than to positive ones, Glosten et al. (1993) defined the TARCH (threshold ARCH) process by allowing $\alpha_+ \neq \alpha_-$ in the equation with $\beta = 1$. (Hence $\alpha_+ < \alpha_-$ generally holds in financial applications.) For a broad overview of the various generalisations of ARCH models and some of their properties we refer the reader to Terasvirta (2009).

Due to the popularity of the quadratic ARCH models in finance, their probabilistic properties are quite well-studied and well-understood. It is a well-known fact for $\beta = 1$ (see e.g. Embrechts et al. (1997)) that not all choices of $(\omega, \alpha_+, \alpha_-)$ and of the distribution of Z_t permit a stationary solution of Eq. (1). For instance, if Z_t is normally distributed, the quadratic ARCH model (i.e. the case $\alpha_+ = \alpha_-$) has a stationary solution if and only if $\alpha_+ = \alpha_- < 2\exp(\delta) \approx 3.562$, where δ is the Euler constant. (A different choice for the noise distribution yields a different domain of stationarity.) Also, much is

* Corresponding author. Tel.: +36 1 327 5630.
E-mail addresses: elekpeti@cs.elte.hu (P. Elek), markus@cs.elte.hu (L. Márkus).

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known about the tail behaviour of the stationary distribution if $\beta = 1$. It was proven two decades ago (Goldie, 1991) that the simple ARCH process has a regularly varying tail (roughly speaking: a polynomially decaying tail) even when Z_t is normally distributed. This phenomenon is often summarised as: "light-tailed input can cause heavy-tailed output". More generally, Borkovec and Klüppelberg (2001) proved that the AR(1) model driven by a quadratic ARCH(1) innovation has regularly varying tail for a very general class of noise distributions. Using the concepts of extreme value theory (EVT) it follows that the stationary distribution of quadratic ARCH processes belongs to the maximum domain of attraction of the Frechet extreme value distribution or, equivalently, their tail can be approximated by a generalised Pareto distribution (GPD) with shape parameter $\xi > 0$. (For an introduction into EVT we refer the reader to Embrechts et al. (1997).)

The $0 < \beta < 1$ case – where the conditional variance is increasing more slowly than a quadratic function of the lagged values – is very different from the usual $\beta = 1$ parameter choice, and is much less studied in the literature. The model may then be called the β-TARCH process and was analysed e.g. by Guegan and Diebolt (1994). It follows relatively easily from the drift condition for Markov chains (Meyn and Tweedie, 1993) that in the $0 < \beta < 1$ case the X_t process defined by (1) is stationary irrespective of the choice for the parameters and for the distribution of Z_t (provided that the latter has a finite second moment and some basic conditions for its density are fulfilled). Moreover, if the mth moment of Z_t is finite, the mth moment of the stationary distribution of X_t will be finite, too (see Guegan and Diebolt (1994), or in a more general setting Elek and Márkus (2008)). Hence, if all moments of Z_t are finite and its distribution has infinite support, the distribution of X_t may only belong to the maximum domain of attraction of the Gumbel law and, equivalently, the shape parameter of the GPD fitted to it may only be zero — if the distribution belongs to the maximum domain of attraction of an extreme value law at all.

This result already yields that the β-TARCH model is lighter tailed than the usual, quadratic specification: for light-tailed Z_t noises the tail of X_t decays faster to zero than a polynomial function. The finding, however, does not determine the exact tail behaviour: the maximum domain of attraction of the Gumbel law contains distributions of many different types (e.g. normally, exponentially or log-normally decaying ones). In this paper we give a more precise estimate for the tail decay by showing that X_t has an approximately Weibull-like tail provided that Z_t has a Weibull-like distribution. Our research is motivated by the fact that β-TARCH models proved useful for modeling conditional heteroscedasticity in areas where the quadratic ARCH model was considered too heavy tailed, such as in the analysis of water discharge series of rivers with large catchments. (See e.g. Elek and Márkus (2008) or in a broader context Szilágyi et al. (2006).)

Throughout the paper we will use the notation $\bar{F}_X(u) = 1 - F_X(u)$ for the survival function and $f_X(u)$ for the density function of the random variable X.

2. Tail behaviour

To examine the tail behaviour of X_t let us introduce an assumption on the tail of Z_t:

Assumption 1. Z_t is an i.i.d. sequence with a symmetric, absolutely continuous probability distribution. Moreover, there exist $u_0 > 0$, $\gamma > 0$, $\kappa > 0$, $K_1 > 0$ and K_2 such that its probability density satisfies

$$f_{Z_t}(u) = K_1|u|^{K_2} \exp(-\kappa|u|^{\gamma}) \tag{2}$$

for every $|u| > u_0$, and $f_{Z_t}(u)$ is bounded away from zero on $[-u_0, u_0]$.

According to this assumption, Z_t has a Weibull-like tail with exponent γ. The Gaussian ($\gamma = 2$) and the Laplace ($\gamma = 1$) distributions are obtained as special cases.

Guegan and Diebolt (1994) showed for under the assumption $\min(\alpha_+, \alpha_-) > 0$ that if $\beta > (\gamma - 1)/\gamma$, X_t has no exponential moment (i.e. it is heavier tailed than the exponential distribution) while if $\beta < (\gamma - 1)/\gamma$, X_t has a moment generating function defined around the neighbourhood of zero. This finding already suggests that X_t may possess (approximately) a Weibull-like tail with exponent $\gamma(1 - \beta)$. Assuming a normally distributed noise (i.e. $\gamma = 2$), $\alpha_+ = \alpha_-$ and $1/2 < \beta < 1$, Robert (2000) argued that this is indeed the case: under his assumptions X_t has a Weibull-like tail with exponent $2(1 - \beta)$. Although the proof of his conjectures seems to be incomplete,[1] some of his ideas are useful for proving that X_t has an approximately Weibull-like tail even if we consider the more general case, i.e. $\alpha_+ \neq \alpha_-$, $\gamma \neq 2$ and $0 < \beta < 1$.

Theorem 1. *Assume that X_t satisfies Eq. (1), Assumption 1 holds, and $\omega > 0$, $\min(\alpha_+, \alpha_-) > 0$, $0 < \beta < 1$. Then, using the notation $\alpha = \max(\alpha_+, \alpha_-)$, the survival function of the stationary distribution of X_t satisfies*

$$\exp\left(-\frac{\alpha^{-\gamma/2}\kappa\gamma\beta^{-\frac{\beta}{1-\beta}}}{2}u^{\gamma(1-\beta)} + O(u^{\gamma(1-\beta)/2})\right) \le \bar{F}_{X_t}(u)$$
$$\le \exp\left(-\frac{(\alpha+\omega)^{-\gamma/2}\kappa\gamma\beta^{-\frac{\beta}{1-\beta}}}{2}u^{\gamma(1-\beta)} + O(u^{\gamma(1-\beta)/2})\right). \tag{3}$$

[1] He derives a functional equation for the logarithm of the moment generating function $L_Y(s) = E(\exp(sX_t))$ of $Y_t = \log X_t^2$ and estimates the tail of Y_t on the basis of the behaviour of $L_Y(s)$ around ∞. During the calculations he assumes (see Appendix 1 of his paper) that if a function g satisfies $g(x) - g(\alpha x) = O(1/x)$ as $x \to \infty$, then $g(x) = O(1/x)$. However, this is not the case: if e.g. $g(x) = \sin(2\pi \log x/\log \alpha)$ then $g(x) - g(\alpha x) = 0$.