



## Fisher information of scale

Peter Ruckdeschel<sup>a,b,\*</sup>, Helmut Rieder<sup>c</sup>

<sup>a</sup> Fraunhofer ITWM, Abt. Finanzmathematik, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany

<sup>b</sup> TU Kaiserslautern, AG Statistik, FB. Mathematik, P.O. Box 3049, 67653 Kaiserslautern, Germany

<sup>c</sup> Mathematical Institute, University of Bayreuth, 95440 Bayreuth, Germany

### ARTICLE INFO

#### Article history:

Received 6 May 2010

Received in revised form 17 August 2010

Accepted 18 August 2010

Available online 21 August 2010

#### MSC:

62F12

62F35

#### Keywords:

One-dimensional scale

Fisher information bound

$L_2$ -differentiability

LAN

Absolute continuity of measures and functions

### ABSTRACT

Motivated by the information bound for the asymptotic variance of M-estimates for scale, we define Fisher information of scale of any distribution function  $F$  on the real line as the supremum of all  $\left( \int x \phi'(x) F(dx) \right)^2 / \int \phi^2(x) F(dx)$ , where  $\phi$  ranges over the continuously differentiable functions with derivative of compact support and where, by convention,  $0/0 := 0$ . In addition, we enforce equivariance by a scale factor. Fisher information of scale is weakly lower semicontinuous and convex. It is finite iff the usual assumptions on densities hold, under which Fisher information of scale is classically defined, and then both classical and our notions agree. Fisher information of finite scale is also equivalent to  $L_2$ -differentiability and local asymptotic normality, respectively, of the scale model induced by  $F$ .

© 2010 Elsevier B.V. All rights reserved.

### 1. Motivation and definition

If  $F$  is any distribution function on  $\mathbb{R}$ , the real line, and  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  a suitable score function such that  $\int \phi dF = 0$ , an  $M$ -estimate of scale  $S_n$  may formally be defined by

$$\sum_{i=1}^n \phi \left( \frac{x_i}{S_n} \right) = 0. \quad (1.1)$$

The estimand refers to the scale model  $(F_\sigma)_{0 < \sigma < \infty}$  induced by  $F = F_1$ , where  $F_\sigma(x) = F(x/\sigma)$ .

Taylor expanding  $\phi(x/s) = \phi(x/\sigma) - (s - \sigma)\phi'(x/\sigma)x/\sigma^2 + \dots$ , we formally obtain

$$\sqrt{n}(S_n - \sigma) = \sigma \frac{n^{-1/2} \sum_{i=1}^n \phi(x_i/\sigma)}{n^{-1} \sum_{i=1}^n \phi'(x_i/\sigma) x_i/\sigma} + \dots \quad (1.2)$$

such that under observations  $x_1, \dots, x_n$  i.i.d.  $\sim F_\sigma$  and assuming sufficient regularity, in particular consistency,  $\sqrt{n}(S_n - \sigma)$  will as  $n \rightarrow \infty$  be asymptotically normal with mean zero and variance

\* Corresponding author at: Fraunhofer ITWM, Abt. Finanzmathematik, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany.

E-mail addresses: [Peter.Ruckdeschel@itwm.fraunhofer.de](mailto:Peter.Ruckdeschel@itwm.fraunhofer.de) (P. Ruckdeschel), [Helmut.Rieder@uni-bayreuth.de](mailto:Helmut.Rieder@uni-bayreuth.de) (H. Rieder).

$$V(\phi, F_\sigma) = \sigma^2 V_1(\phi, F), \quad V_1(\phi, F) := \frac{\int \phi^2(x) F(dx)}{\left(\int x\phi'(x) F(dx)\right)^2}. \quad (1.3)$$

If  $\phi$  is differentiable with continuous derivative of compact support, both  $\phi(x)$  and  $x\phi'(x)$  are bounded, so the integrals in (1.3) are well defined for any distribution  $F$  on the Borel  $\sigma$ -algebra  $\mathbb{B}$  of  $\mathbb{R}$ . As in the theory of generalized functions (Rudin, 1991, Ch. 6), regularity conditions are shifted to the test functions whenever possible.

The usual information bound for asymptotic variance would say that  $V(\phi, F_\sigma) \geq \mathcal{I}_s^{-1}(F_\sigma)$  and, hopefully, the lower bound will also be achieved.

This leads us to the following definition of  $\mathcal{I}_{s1}(F)$ . The extension to  $\mathcal{I}_s(F_\sigma)$  for the scale transforms  $F_\sigma$  of  $F$  matches (1.3).

**Definition 1.1.** Fisher information of scale, for any distribution  $F$  on the real line, is defined by

$$\mathcal{I}_{s1}(F) := \sup_{\phi \in \mathcal{C}_{c1}} \frac{\left(\int x\phi'(x) F(dx)\right)^2}{\int \phi^2(x) F(dx)}, \quad (1.4)$$

where  $\mathcal{C}_{c1}$  denotes the set of all differentiable functions  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  whose derivative is continuous and of compact support, and  $0/0 := 0$  by convention. For the scale transforms  $F_\sigma$  of  $F$  we define

$$\mathcal{I}_s(F_\sigma) := \sigma^{-2} \mathcal{I}_{s1}(F), \quad 0 < \sigma < \infty. \quad (1.5)$$

**Remark 1.2.** Since the map  $\phi \mapsto \phi_\sigma$ , where  $\phi_\sigma(x) := \phi(\sigma x)$  and  $\phi'_\sigma(x) = \sigma \phi'(\sigma x)$ , defines a one-to-one correspondence on  $\mathcal{C}_{c1}$ , we obtain the scale invariance of  $\mathcal{I}_{s1}$ ,

$$\mathcal{I}_{s1}(F_\sigma) = \mathcal{I}_{s1}(F), \quad 0 < \sigma < \infty. \quad (1.6)$$

So extension (1.5) is needed to obtain scale equivariance. In the scale model, as opposed to location, it matters whether a given distribution  $F$  is considered element  $F = F_1$  or, for example, element  $F = F_{.5}$  (in the scale model generated by  $F_2$ ).  $\square$

Motivated by the information bound, Definition 1.1 is intrinsically statistical. It does not a priori use the assumption of, and suitable conditions on, densities. These properties rather follow from the definition in case  $\mathcal{I}_s$  is finite. Another advantage is that Definition 1.1 implies certain topological properties (convexity and lower continuity) of  $\mathcal{I}_s$ .

The definition parallels (Huber, 1981, Def. 4.1) in the location case,

$$\mathcal{I}_1(F) := \sup_{\phi} \frac{\left(\int \phi'(x) F(dx)\right)^2}{\int \phi^2(x) F(dx)}, \quad (1.7)$$

where  $\phi$ , subject to  $\int \phi^2 dF > 0$ , ranges over the (smaller) set  $\mathcal{C}_c^1$  of all continuously differentiable functions which themselves are of compact support.  $\mathcal{I}_1$  is shift invariant.

Huber (1981, p. 79) states vague lower semicontinuity and convexity of  $\mathcal{I}_1$ . By Huber (1981, Thm. 4.2),  $\mathcal{I}_1(F)$  is finite iff  $F$  is absolutely continuous with an absolutely continuous density  $f$  such that  $f'/f \in L_2(F)$ , in which case  $\mathcal{I}_1(F) = \int (f'/f)^2 dF$ .

**Remark 1.3.** The latter result, by arguments of the proof to Theorem 2.2, is still obtained if definition (1.7) is based on  $\mathcal{C}_{c1}$ . Only vague lower semicontinuity of  $\mathcal{I}_1$  would be weakened to weak continuity (which, however, makes no difference in the setup of normed measures). The convention  $0/0 := 0$  could replace the side condition  $\phi \neq 0$  a.e.  $F$  in (1.7) as well.

The non-suitability of  $\mathcal{C}_c^1$ , and suitability of  $\mathcal{C}_{c1}$  instead, is the tribute to the scale model, for which the functions  $x \mapsto x\phi'(x)$  need to be dense in  $L_1(F_0)$  with respect to the punctuated (substochastic) measure  $F_0$  introduced in (2.1).  $\square$

Fisher information of scale has been treated by Huber (1964, 1981) not in the previous generality but only under suitable assumptions on densities and, in an auxiliary way, has been reduced to the location case by symmetrization and the log-transform (Huber, 1981, Sec. 5.6).

## 2. Main results

**Proposition 2.1.**  $\mathcal{I}_{s1}$  is weakly lower semicontinuous and convex.

Zero observations do not contain any information about scale. Removing the mass of any distribution  $F$  at zero, we define the punctuated, possibly substochastic measure  $F_0$  by

$$F_0 := F - F(\{0\})1_0, \quad (2.1)$$

where  $1_0$  denotes Dirac measure at 0. In terms of distribution functions, denoting by  $1_{[0,\infty)}$  the indicator function, we have  $F_0(x) = F(x) - (F(0) - F(0-))1_{[0,\infty)}(x)$ .

**Theorem 2.2.** For any distribution  $F$  on the real line,  $\mathcal{I}_{s1}(F)$  is finite iff

Download English Version:

<https://daneshyari.com/en/article/1152736>

Download Persian Version:

<https://daneshyari.com/article/1152736>

[Daneshyari.com](https://daneshyari.com)