Statistics and Probability Letters 84 (2014) 1–8

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Covariance operator estimation of a functional autoregressive process with random coefficients

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ARTICLE INFO

Article history:
Received 17 January 2013
Received in revised form 16 September 2013
Accepted 16 September 2013
Available online 25 September 2013

MSC:
62G05
62G07

Keywords:
Functional autoregressive process with random coefficients
Covariance operator
Eigenvalues and eigenvectors
Hilbert–Schmidt norm

ABSTRACT

We deal with the covariance and cross covariance operators estimation of a Hilbert space valued autoregressive process with random coefficients. We establish bounds for empirical estimators in mean square error and almost sure convergence in Hilbert–Schmidt norm. Consistent estimators of the eigenvalues are also derived.

1. Introduction

In recent years, technological advances in collecting and storing data from many fields such as economy, finance, industry, biology, medicine etc., have made statisticians consider them as "high dimensional" or of functional nature. Then the data may be considered as a sample of a continuous time series so need to be handled as an observation of a valued function space r.v's. This field is known as Functional Data Analysis (FDA) and has been well developed in many textbooks (see Bosq (2000), Bosq and Blanke (2007), Ferraty and Vieu (2006) and Ramsay and Silverman (2005) and references therein). Particularly, as a modeling framework we may cite Functional Autoregressive Processes with deterministic operator. This class has been well investigated by many authors and is the object of various applications (see Bosq (2000) and Bosq and Blanke (2007) and references therein).

In this paper we are interested in Functional Autoregressive Processes with random coefficients. The random coefficients time series models are used as a tool for handling possible nonlinear features of real life data as explained for example in Nicholls and Quinn (1982) and Tjøsteim (1994). Moreover estimation theory has been carried out by many authors. Among them, we cite Tjøsteim (1986, 1994) who has studied a simple doubly stochastic model where the parameter of AR process is considered as a Markov chain (θ_t) with values in a finite state space or well (θ_t) is itself an AR process. If the rv's (θ_t) are i.i.d. the model is said to be a random coefficient autoregressive model (RCA model). Autoregressive models with Markov regimes or a hidden Markov chain are well studied and useful in a variety of situations (see Brandt (1986), Hamilton (1990) and Holst et al. (1994)).

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0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.09.018

Our main goal in this paper is the estimation of the covariance and cross covariance operator of functional autoregressive processes with random coefficients. Our results generalize those in Guillas (2002). The interest of these asymptotic results covers a wide scope of problems, including for instance the prediction of time continuous processes (see Allam (2007), Bosq (2000), Bosq and Blanke (2007), Ferraty and Vieu (2006) and Ramsay and Silverman (2005), testing procedures for classification curves (see Ferraty et al. (2007) and functional principal components analysis (see Hall and Hosseini-Nasab (2006), for the latest developments on FPCA). As a byproduct, because FPCA also plays a major role in regression analysis we can refer to Ferraty et al. (2012) for recent advances on functional linear model, and to Ferraty and Vieu (2006) for nonparametric functional regression. On a semi-parametric functional regression setting one can see Chen et al. (2011) for functional single index regression, Ferraty et al. (2013) for the test for functional projection pursuit regression, Aneiros and Vieu (2006) and Ferraty et al. (2002) for partial linear functional regression, our results will also be of interest in this area.

In the sequel we introduce the model and define the estimators and give bounds in mean square error in Hilbert–Schmidt norm. Then we deduce their almost sure convergence with the same rates given in the valued Hilbert space autoregressive processes case (see Bosq (2000) Chap. 6).

The paper is organized as follows: Section 2 presents notations and definitions. In Section 3, we give the main results, and Section 4 contains the proofs.

2. Notations and definitions

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and $(H, \|.\|)$ a real separable Hilbert space with inner product $\langle ., . \rangle_H$ and endowed with its Borel σ-field $\mathcal{H}$. We denote by $\mathcal{L}(H)$ the space of linear bounded operators defined over H into H, and by $\|.\|_{\mathcal{L}}$ the usual norm of linear bounded operators. A H-valued sequence $(\varepsilon_n, n \in \mathbb{Z})$ of random variables defined on $(\Omega, \mathcal{A}, P)$, with values in H is said to be a H-valued white noise if $\varepsilon_n, n \in \mathbb{Z}$, are independent identically distributed of zero mean and $E\|\varepsilon_n\|^2 := \sigma_\varepsilon^2 > 0$. Let $(\rho_n, n \in \mathbb{Z})$ be a measurable sequence of random operators defined on $(\Omega, \mathcal{A}, P)$ with values in $\mathcal{L}(H)$ endowed with its Borel σ-field. We consider a process $(X_n, n \in \mathbb{Z})$ defined on $(\Omega, \mathcal{A}, P)$ with values in $(H, \mathcal{H})$ satisfying the following equation:

$$X_n = \rho_n X_{n-1} + \varepsilon_n, \quad n \in \mathbb{Z} \tag{1}$$

$(X_n, n \in \mathbb{Z})$ is said to be a Hilbertian random coefficients autoregressive process ($HRCA$).

In the sequel we assume the following condition:

$\mathbf{C}_1$:

- The rv's $(\rho_n, n \in \mathbb{Z})$ are i.i.d.
- The two sequences $(\rho_n, n \in \mathbb{Z})$ and $(\varepsilon_n, n \in \mathbb{Z})$ are independent.
- $E\|X_0\|^4 < \infty$.
- $\sup_i \|\rho_i\|_{\mathcal{L}(H)} < 1$ *a.s.*

Notice that under the condition $\mathbf{C}_1$, the Eq. (1) has a unique stationary solution (Allam, 2007; Mourid, 2004). At first we recall some known facts (see Bosq (2000) Chap. 1.5). For a H-valued rv's X with a zero mean and $E\|X\|^2 < \infty$, the covariance operator C_X of X is defined by:

$$C_X(x) = E(\langle X, x\rangle X), \quad x \in H.$$

The operator C_X is symmetric, compact, positive and nuclear. The cross covariance operator of the rv's X and Y, a H-valued Hilbert space rv's with $E(Y) = 0$ and $E\|Y\|^2 < \infty$, is defined by

$$C_{X,Y}(x) = E(\langle X, x\rangle Y), \quad x \in H.$$

The operator $C_{X,Y}$ is compact with adjoint operator $C^*_{X,Y} = C_{Y,X}$.

A linear operator T defined over H is Hilbert–Schmidt if:

$$\|T\|_S^2 := \sum_{i=1}^{\infty} \|Tf_i\|^2 = \sum_{i,j=1}^{\infty} \langle Tf_i, f_j\rangle_H^2 < \infty$$

where $(f_i, i \in \mathbb{N})$ is an arbitrary orthonormal basis in H and $\|.\|_S$ is called the Hilbert–Schmidt norm. We denote by $\mathcal{S}$ the space of Hilbert–Schmidt operators over H. It is a separable Hilbert space with the inner product: for $T_1, T_2 \in \mathcal{S}$,

$$\langle T_1, T_2\rangle_S = \sum_{i,j=1}^{\infty} \left\langle T_1 f_i, f_j\right\rangle_H \left\langle T_2 f_i, f_j\right\rangle_H .$$

3. Main results

Suppose we observe $(X_1, X_1, \ldots, X_n)$ satisfying (1) under the condition $\mathbf{C}_1$. A natural estimator of C_{X_0} is the empirical covariance operator:

$$C_n(x) = \frac{1}{n} \sum_{i=1}^{n} \langle X_i, x\rangle X_i, \quad x \in H$$