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When is a Markov chain regenerative?

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ABSTRACT

A sequence of random variables $\{X_n\}_{n\geq 0}$ is called *regenerative* if it can be broken up into iid components. The problem addressed in this paper is that of determining under what conditions a Markov chain is regenerative. It is shown that an irreducible Markov chain with a countable state space is regenerative for any initial distribution if and only if it is *recurrent* (*null or positive*). An extension of this to the general state space case is also discussed.

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1. Introduction

A sequence of random variables $\{X_n\}_{n\geq 0}$ is *regenerative* (see the formal definition below) if it can be broken up into iid components. This makes it possible to apply the laws of large numbers for iid random variables to such sequences. In particular, in Athreya and Roy (2012) this idea was used to develop Monte Carlo methods for estimating integrals of functions with respect to distributions π that may or may *not* be proper (that is, $\pi(S)$ can be ∞ where S is the underlying state space). A regenerative sequence $\{X_n\}_{n\geq 0}$, in general, need not be a Markov chain. In this short work we address the question of when a Markov chain is regenerative.

We now give a formal definition of a regenerative sequence of random variables.

Definition 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(S, \mathcal{S})$ be a measurable space. A sequence of random variables $\{X_n\}_{n\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ with values in $(S, \mathcal{S})$ is called *regenerative* if there exists a sequence of (random) times $0 < T_1 < T_2 < \cdots$ such that the excursions $\{X_n : T_j \leq n < T_{j+1}, \tau_j\}_{j\geq 1}$ are iid where $\tau_j = T_{j+1} - T_j$ for $j = 1, 2, \ldots,$ that is,

$$P(\tau_j = k_j, X_{T_j+q} \in A_{q,j},\ 0 \leq q < k_j, j = 1, \ldots, r) = \prod_{j=1}^{r} P(\tau_1 = k_j, X_{T_1+q} \in A_{q,j}, 0 \leq q < k_j),$$

for all $k_1, \ldots, k_r \in \mathbb{N}$, the set of positive integers, $A_{q,j} \in \mathcal{S}$, $0 \leq q < k_j, j = 1, \ldots, r$, and $r \geq 1$. The random times $\{T_n\}_{n\geq 1}$ are called *regeneration times*.

From the definition of a regenerative sequence and the strong law of large numbers the next remark follows and Athreya and Roy (2012) used it as a basis for constructing Monte Carlo estimators for integrals of functions with respect to improper distributions.

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Remark 1. Let $\{X_n\}_{n\geq 0}$ be a regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$. Let

$$\pi(A) = E\left(\sum_{j=T_1}^{T_2-1} I_A(X_j)\right) \quad \text{for } A \in \mathcal{S}. \tag{1}$$

(The measure π is known as the canonical (or occupation) measure for $\{X_n\}_{n\geq 0}$.) Let $N_n = k$ if $T_k \leq n < T_{k+1}$, $k, n = 1, 2, \ldots$. Suppose $f, g : S \to \mathbb{R}$ to be two measurable functions such that $\int_S |f|d\pi < \infty$ and $\int_S |g|d\pi < \infty$, $\int_S g d\pi \neq 0$. Then, as $n \to \infty$:

(i) (regeneration estimator)

$$\hat{\lambda}_n = \frac{\sum_{j=0}^{n} f(X_j)}{N_n} \xrightarrow{\text{a.s.}} \int_S f d\pi,$$

(ii) (ratio estimator)

$$\hat{R}_n = \frac{\sum_{j=0}^{n} f(X_j)}{\sum_{j=0}^{n} g(X_j)} \xrightarrow{\text{a.s.}} \frac{\int_S f d\pi}{\int_S g d\pi}.$$

Athreya and Roy (2012) showed that if π happens to be improper then the standard time average estimator, $\sum_{j=0}^{n} f(X_j)/n$, based on Markov chains $\{X_n\}_{n\geq 0}$ with π as its stationary distribution will converge to zero with probability 1, and hence is not appropriate. On the other hand, the *regeneration* and *ratio* estimators, namely $\hat{\lambda}_n$ and $(\int_S g d\pi)\hat{R}_n$ (assuming $\int_S g d\pi$ is known), defined above produce strongly consistent estimators of $\int_S f d\pi$ and they work whether π is proper or improper. Regeneration methods have been proved to be useful in a number of other statistical applications. For example, in the case of a proper target, that is when $\pi(S)$ is finite, a regenerative method has been used to construct consistent estimates of the asymptotic variance of Markov chain Monte Carlo (MCMC) based estimates (Mykland et al., 1995), to obtain quantitative bounds on the rate of convergence of Markov chains (see e.g. Roberts and Tweedie, 1999), and to construct bootstrap methods for Markov chains (see e.g. Datta and McCormick, 1993; Bertail and Clémençon, 2006). The regeneration method has also been used in nonparametric estimation for null recurrent time series (Karlsen and Tjøstheim, 2001).

2. The results

Our first result shows that a necessary and sufficient condition for an irreducible countable state space Markov chain to be regenerative for any initial distribution is that it is recurrent.

Theorem 1. *Let $\{X_n\}_{n\geq 0}$ be a time homogeneous Markov chain with a countable state space $S = \{s_0, s_1, s_2, \ldots\}$. Let $\{X_n\}_{n\geq 0}$ be irreducible, that is, for all $s_i, s_j \in S$, $P(X_n = s_j$ for some $1 \leq n < \infty \mid X_0 = s_i) > 0$. Then for any initial distribution of X_0, the Markov chain $\{X_n\}_{n\geq 0}$ is regenerative if and only if it is recurrent, that is, for all $s_i \in S$, $P(X_n = s_i$ for some $1 \leq n < \infty \mid X_0 = s_i) = 1$.*

The next result gives a sufficient condition for a Markov chain $\{X_n\}_{n\geq 0}$ with countable state space to be regenerative.

Theorem 2. *Let $\{X_n\}_{n\geq 0}$ be a time homogeneous Markov chain with a countable state space $S = \{s_0, s_1, s_2, \ldots\}$. Suppose there exists a state $s_{i_0} \in S$ such that*

$$P(X_n = s_{i_0} \text{ for some } 1 \leq n < \infty | X_0 = s_{i_0}) = 1, \tag{2}$$

that is, s_{i_0} is a recurrent state. Then $\{X_n\}_{n\geq 0}$ is regenerative for any initial distribution ν of X_0 such that $P(X_n = s_{i_0}$ for some $1 \leq n < \infty \mid X_0 \sim \nu) = 1$.

Remark 2. It is known that a necessary and sufficient condition for (2) to hold is that $\sum_{n=1}^{\infty} P(X_n = s_{i_0} \mid X_0 = s_{i_0}) = \infty$ (see e.g. Athreya and Lahiri, 2006, Corollary 14.1.10).

Next we give a necessary condition for a Markov chain $\{X_n\}_{n\geq 0}$ with countable state space to be regenerative.

Theorem 3. *Let $\{X_n\}_{n\geq 0}$ be a time homogeneous Markov chain with a countable state space $S = \{s_0, s_1, s_2, \ldots\}$ and initial distribution ν of X_0. Suppose $\{X_n\}_{n\geq 0}$ is regenerative. Then, there exists a nonempty set $S_0 \subset S$ such that for all $s_i, s_j \in S_0$, $P(X_n = s_j$ for some $1 \leq n < \infty \mid X_0 = s_i) = 1$.*

Remark 3. Under the hypothesis of Theorem 3, the Markov chain $\{X_n\}_{n\geq 0}$ is regenerative for any initial distribution of X_0 such that $P(X_n \in S_0$ for some $1 \leq n < \infty) = 1$.