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Fractional Poisson processes and their representation by infinite systems of ordinary differential equations[☆]

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ABSTRACT

Fractional Poisson processes, a rapidly growing area of non-Markovian stochastic processes, are useful in statistics to describe data from counting processes when waiting times are not exponentially distributed. We show that the fractional Kolmogorov–Feller equations for the probabilities at time t can be represented by an infinite linear system of ordinary differential equations of first order in a transformed time variable. These new equations resemble a linear version of the discrete coagulation–fragmentation equations, well-known from the non-equilibrium theory of gelation, cluster-dynamics and phase transitions in physics and chemistry.

1. Introducing fractional Poisson processes

Since the late 1990s there has been a great interest in non-Markovian continuous-time processes, especially those arising from waiting times between two events that are not exponentially distributed (for a general overview see Embrechts et al. (1997) and Grandell (1997)) but sub-exponentially, for example (Jumarie, 2001; Laskin, 2003)

$$\mathrm{prob}(T_w < t) = 1 - E_\beta(-\lambda t^\beta) \tag{1}$$

where $\mathrm{prob}(T_w < t)$ is the probability that the waiting time T_w is less than some t, $\lambda > 0$, $0 < \beta \le 1$ and E_β is the Mittag-Leffler function. The Mittag-Leffler function in Eq. (1), has the series representation $E_\beta(z) = \sum_{m=0}^{\infty} z^m/\Gamma(\beta m + 1)$ and is a fractional generalization of the exponential function; where $\Gamma(x) = \int_0^\infty s^{x-1}\exp(-s)ds$ is the Gamma function (for $\beta = 1$ the exponential function is recovered).

Starting from the waiting time distribution Eq. (1), Laskin (2003) introduced the fractional Poisson process as the counting process with probability $P_\beta(n, t)$ of n items ($n = 0, 1, 2, \ldots$) arriving by a time t. Beghin and Orsingher (2009) pointed out, that the approach of Laskin (2003) is equivalent to solving their fractional equations

$$\frac{d^\beta}{dt^\beta} P_\beta(n, t) = \lambda(P_\beta(n-1, t) - P_\beta(n, t)), \quad 0 < \beta \le 1, \tag{2}$$

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where the fractional derivative is taken in the sense of Dzerbayshan–Caputo (e.g. Podlubny (1999, p. 78)), and is defined on a twice continuously differentiable function $f(t)$ as the usual derivative for $\beta = 1$ and for $0 < \beta < 1$ is

$$\frac{d^\beta}{dt^\beta} f(t) \equiv \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{ds}{(t-s)^\beta} \frac{d}{ds} f(s). \tag{3}$$

Their solution with the initial condition $P_\beta(n, t = 0) = \delta_{n,0}$ is given by

$$P_\beta(n, t) = (-1)^n \sum_{j=n}^{\infty} \binom{j}{n} \frac{(-1)^j (\lambda t^\beta)^j}{\Gamma(\beta j + 1)} = (-1)^n \sum_{j=n}^{\infty} \binom{j}{n} \phi_\beta(j, t) \tag{4}$$

where the functions $\phi_\beta(j, t)$ are defined as

$$\phi_\beta(j, t) \equiv (-1)^j \frac{(\lambda t^\beta)^j}{\Gamma(\beta j + 1)}, \quad j = 0, 1, 2, \ldots. \tag{5}$$

Note that for $n = 0$ in Eq. (4) we recover the Mittag-Leffler function with index β, $E_\beta(-\lambda t^\beta)$. This is not surprising as $P_\beta(0, t)$ is the probability of no birth taking place up to time t, i.e. prob$(T_w > t)$ which is nothing else but the complement to the waiting time probability, Eq. (1). Furthermore, one readily recovers the normalization condition $\sum_{n=0}^{\infty} P_\beta(n, t) = 1$.

Eq. (2) together with Eq. (3) are now considered the standard fractional Kolmogorov–Feller equations for a fractional Poisson process. They are the basis of a fast growing branch of probability theory (e.g. Beghin and Orsingher (2009, 2010), Beghin and Macci (2013) and Orsingher and Polito (2013)).

We shall show in this letter that the fractional Poisson process Eq. (2) can also be described by an infinite linear system of ordinary differential equations (ODE) of first order in a transformed time variable for the probabilities $P_\beta(n, t)$ on the left-hand side, and on the right-hand side we have the usual two terms of a standard Poisson process plus infinitely many more terms consisting of some time-independent constants times $P_\beta(m, t)$, and $m > n$. This result, formulated in Theorem 3.1, Eq. (8), is what we term the "ODE-representation of the Kolmogorov–Feller equations".

These new equations for the fractional Poisson process bear a striking resemblance to the linear version of the discrete cluster equations for the Glauber kinetic Ising model, as discussed in Binder and Müller-Krumbhaar (1974) and Kreer (1993) which is briefly discussed in Appendix B, Eq. (B.3). These cluster ODEs describe a typical dynamics in which clusters consisting of n particles, say, can coagulate with other clusters to form larger clusters or fragment to form smaller ones. Thus, our approach allows us, in principle, to understand the dynamics of the fractional Poisson process in terms of cluster interactions. Consequently, our "ODE-representation of the Kolmogorov–Feller equations" belongs to a wider class of coagulation–fragmentation equations, which are of major interest to the wider community, as they show a variety of interesting features, such as asymptotic self-similarity for large times ("dynamical scaling"), gelating at finite time, metastability, etc. They may also be a tool to understand phenomena of non-equilibrium statistical physics and phase transitions. For further reading we refer to Spouge (1984), Ball and Carr (1990), Kreer (1993), da Costa (1995), Laurençot and Mischler (2002) and McBride et al. (2010).

One of the aims of this letter is to bring the community of probabilists dealing with fractional Poisson processes and the community of analysts and physicists dealing with coagulation–fragmentation equations closer as both subjects seem to be more related than previously thought.

The structure of the paper is as follows: Before presenting our main result in Section 3 we prove in Section 2 some necessary Lemma dealing with a combinatorial inversion formula for certain functions.

2. An application of Krattenthaler's theorem

To prove an essential Lemma, we heuristically define "infinite" vectors from the solutions Eqs. (4) and (5) of the fractional Poisson process,

$$[P_\beta(0, t), -P_\beta(1, t), \ldots, (-1)^n P_\beta(n, t), \ldots]^T$$
$$\text{and} \quad [\phi_\beta(0, t), \phi_\beta(1, t), \ldots, \phi_\beta(n, t), \ldots]^T.$$

We may then write the solutions Eq. (4) formally as follows

$$\begin{bmatrix} P_\beta(0, t) \\ -P_\beta(1, t) \\ P_\beta(2, t) \\ -P_\beta(3, t) \\ P_\beta(4, t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \phi_\beta(0, t) \\ \phi_\beta(1, t) \\ \phi_\beta(2, t) \\ \phi_\beta(3, t) \\ \phi_\beta(4, t) \\ \vdots \end{bmatrix}, \tag{6}$$