Statistics and Probability Letters 84 (2014) 40–47

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Rates of convergence of extremes from skew-normal samples

Xin Liao [a], Zuoxiang Peng [a], Saralees Nadarajah [b,*], Xiaoqian Wang [c]

[a] School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China
[b] School of Mathematics, University of Manchester, Manchester, United Kingdom
[c] Institute of Mathematics, School of Mathematics Sciences, Nanjing Normal University, Nanjing, 210097, China

ARTICLE INFO

Article history:
Received 2 September 2013
Received in revised form 14 September 2013
Accepted 22 September 2013
Available online 26 September 2013

MSC:
primary 62E20
60G70
secondary 60F15
60F05

Keywords:
Extreme value distribution
Maximum
Rate of convergence
Skew-normal distribution

ABSTRACT

For a skew-normal random sequence, convergence rates of the distribution of its partial maximum to the Gumbel extreme value distribution are derived. The asymptotic expansion of the distribution of the normalized maximum is given under an optimal choice of norming constants. We find that the optimal convergence rate of the normalized maximum to the Gumbel extreme value distribution is proportional to $1/\log n$.

1. Introduction

The biggest weakness of the normal distribution is its inability to model skewed data. This has led to several skewed extensions of the normal distribution. The most popular and the most studied of these extensions is the skew-normal distribution due to Azzalini (1985). A random variable X is said to have a standard skew-normal distribution with shape parameter $\lambda \in \mathbb{R}$ (written as $X \sim \mathrm{SN}(\lambda)$) if its probability density function (pdf) is

$$f_\lambda(x) = 2\phi(x)\Phi(\lambda x), \quad -\infty < x < +\infty,$$

where $\phi(\cdot)$ denotes the standard normal pdf and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function (cdf). It is known that $\mathrm{SN}(0)$ is a standard normal random variable.

The skew-normal distribution has received more applications than any other extension of the normal distribution. Its applications are too many to list. Some applications of the skew-normal distribution that have appeared in the past year alone include: the distribution of threshold voltage degradation in nanoscale transistors by using reaction–diffusion and percolation theory (Islam and Alam, 2011); population structure of Schima superba in Qingliangfeng National Nature Reserve (Liu et al., 2011); rain height models to predict fading due to wet snow on terrestrial links (Paulson and Al-Mreri, 2011); modeling of seasonal rainfall in Africa (Siebert and Ward, 2011); modeling of HIV viral loads (Bandyopadhyay et al., 2012);

* Corresponding author. Tel.: +44 01614732941.
E-mail address: mbbsssn2@manchester.ac.uk (S. Nadarajah).

0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.09.027

multisite flooding hazard assessment in the Upper Mississippi River (Ghizzoni et al., 2012); modeling of diabetic macular Edema data (Mansourian et al., 2012); risks of macroeconomic forecasts (Pinheiro and Esteves, 2012); modeling of current account balance data (Saez et al., 2012); automated neonatal EEG classification (Temko et al., 2012).

The aim of this note is to establish the convergence rate of the distribution of the maxima for samples obeying SN(λ). Chang and Genton (2007) showed that SN(λ) belongs to the domain of attraction of the Gumbel extreme value cdf $\Lambda(x) = \exp\{-\exp(-x)\}$. Rates of convergence of the distribution of maxima for a sequence of independent $SN(0)$ random variables were studied by Hall (1979), Leadbetter et al. (1983) and Nair (1981). Precisely speaking, Leadbetter et al. (1983) proved that

$$\Phi^n(\alpha_n x + \beta_n) - \Lambda(x) \sim \frac{e^{-x}\exp\left(-e^{-x}\right)}{16}\frac{(\log\log n)^2}{\log n}$$

holds for large n with normalized constants α_n and β_n given by

$$\alpha_n = (2\log n)^{-\frac{1}{2}} \quad \text{and} \quad \beta_n = \alpha_n^{-1} - \frac{\alpha_n}{2}(\log\log n + \log 4\pi).$$

The optimal uniform convergence rate of $\Phi^n\left(\widetilde{a}_n x + \widetilde{b}_n\right)$ to $\Lambda(x)$ due to Hall (1979) is

$$\frac{\mathbb{C}_1}{\log n} < \sup_{x\in\mathbb{R}}\left|\Phi^n\left(\widetilde{a}_n x + \widetilde{b}_n\right) - \Lambda(x)\right| < \frac{\mathbb{C}_2}{\log n}$$

for some absolute constants $0 < \mathbb{C}_1 < \mathbb{C}_2$ with normalized constants $\widetilde{a}_n$ and $\widetilde{b}_n$ determined by

$$2\pi\widetilde{b}_n^2\exp\left(\widetilde{b}_n^2\right) = n^2, \qquad \widetilde{a}_n = \widetilde{b}_n^{-1}.$$

The following more informative result was established by Nair (1981):

$$\bar{b}_n^2\left[\bar{b}_n^2\left(\Phi^n\left(\bar{a}_n x + \bar{b}_n\right) - \Lambda(x)\right) - \bar{\kappa}(x)\Lambda(x)\right] \to \left(\bar{\varpi}(x) + \frac{\bar{\kappa}^2(x)}{2}\right)\Lambda(x),$$

where the normalized constants $\bar{a}_n$ and $\bar{b}_n$ are given by

$$1 - \Phi\left(\bar{b}_n\right) = n^{-1}, \qquad \bar{a}_n = \bar{b}_n^{-1},$$

where $\bar{\kappa}(x)$ and $\bar{\varpi}(x)$ are defined as

$$\bar{\kappa}(x) = 2^{-1}\left(x^2 + 2x\right)e^{-x} \quad \text{and} \quad \bar{\varpi}(x) = -8^{-1}\left(x^4 + 4x^3 + 8x^2 + 16x\right)e^{-x}.$$

The contents of this note are organized as follows. Section 2 derives some preliminary results related to SN(λ) like Mills inequalities, Mills ratios, and the distributional tail representation of SN(λ) for $\lambda \neq 0$. Convergence rates of the distribution of the maxima for SN(λ) samples and related proofs are given in Section 3. In the sequel we shall assume that the shape parameter $\lambda \neq 0$.

2. Preliminary results

In this section, some preliminary but important properties about SN(λ) are derived. These properties not only imply that SN(λ) belongs to the max-domain of attraction of the Gumbel extreme value distribution but they also help us to find two pairs of norming constants.

The following Mills inequality and Mills ratio about SN(0) due to Mills (1926) are needed in this section, i.e.,

$$x^{-1}\left(1 + x^{-2}\right)^{-1}\phi(x) < 1 - \Phi(x) < x^{-1}\phi(x) \tag{2.1}$$

for all $x > 0$ and

$$\frac{1 - \Phi(x)}{\phi(x)} \sim \frac{1}{x} \tag{2.2}$$

as $x \to +\infty$. For some improved Mills inequalities, see Mitrinović and Vasić (1970, pp. 177–180) and references therein.

First, we derive Mills inequalities and Mills ratios of SN(λ), which are stated as follows.

Proposition 1. *Let $F_\lambda(x)$ and $f_\lambda(x)$ denote, respectively, the cdf and the pdf of SN(λ). For all $x > 0$, we have*

(i) *if $\lambda > 0$,*

$$x^{-1}\left(1 + x^{-2}\right)^{-1} < \frac{1 - F_\lambda(x)}{f_\lambda(x)} < x^{-1}\left(1 - \frac{\phi(\lambda x)}{\lambda x}\right)^{-1}, \tag{2.3}$$

which implies

$$\frac{1 - F_\lambda(x)}{f_\lambda(x)} \sim \frac{1}{x} \tag{2.4}$$

as $x \to +\infty$;