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Small deviations of the determinants of random matrices with Gaussian entries☆

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ABSTRACT

The probability of small deviations of the determinant of the matrix AA^T is estimated, where A is an $n \times \infty$ random matrix with centered entries having the joint Gaussian distribution. The inequality obtained is sharp in a sense.

1. Introduction and main results

We discuss the problem of estimating probabilities of small deviations for the determinants of random matrices of a special type. The need for the result of this kind emerged during obtaining the asymptotic expansion for the distributions of canonical V-statistics of the third order (Borisov and Volodko, submitted for publication). Moreover, the problem itself seems to be of interest.

The approach of the paper mentioned is based on the representation of a canonical kernel as a special multiple series (this construction was used by Borisov and Volodko, 2008 and Rubin and Vitale, 1980) and on the interpretation of a canonical V-statistic as a smooth functional of the normalized sum of independent random elements from the corresponding Banach space and using available results of Borisov and Solov'ev (1992) for this case. Adduce several fragments from the unpublished work of Borisov and Volodko which show the connection between asymptotic expansions for V-statistics and the present problem. As was mentioned above, we consider a canonical statistic as a smooth functional.

Let $F(x)$ be a measurable functional on some Banach space, satisfying certain regularity conditions (Borisov and Solov'ev, 1992). The most complicated restriction for the verification is the following condition of the so-called stochastic separation from zero of the second Frechet derivative:

$$\sup_z z^{-M}\mathbb{P}(D(\zeta, \eta) \le z) \le B \tag{1.1}$$

for some B and M large enough, where

$$D(x, y) = \mathbb{E}(F^{(2)}(x)[\xi_1, y])^2.$$

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Here ξ_1 is a random sub-Gaussian element with known distribution and values from the Banach space mentioned, ζ and η, are independent corresponding Gaussian elements. For the case of a canonical V-statistic of the third order condition (1.1) can be written as follows:

$$\mathbf{P}\left(\sum_{i=1}^{m}\left(\sum_{j,k=1}^{\infty} f_{ijk}\zeta_j\eta_k\right)^2 < \varepsilon^2\right) \le C(m)\varepsilon^{\gamma} \tag{1.2}$$

for arbitrary large m, where the constant $\gamma > 0$ does not depend on m and ε, $\{\zeta_i\}$ and $\{\eta_k\}$ are independent sequences of independent $N(0,1)$-distributed random variables, $\{f_{ijk}\}$ are absolutely summable coefficients of the kernel representation as a special multiple series (for more details see Borisov and Volodko, 2008).

Denote

$$A_i(j) = \sum_{k=1}^{\infty} f_{ijk}\eta_k.$$

Obviously, the sequence $\{\zeta_k\}$ does not depend on $\{A_i(j)\}$. Consider a conditional probability $\mathbf{P}_\eta$ of the event in (1.2) given $A_i(j)$ (or given vector η):

$$\mathbf{P}_\eta\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{\infty} A_i(j)\zeta_j\right)^2 < \varepsilon^2\right) \le \mathbf{P}_\eta\left(\bigcap_{i=1}^{m}\left\{\left|\sum_{j=1}^{\infty} A_i(j)\zeta_j\right| < \varepsilon\right\}\right) \quad \text{a.s.}$$

Introduce into consideration an $m \times \infty$ matrix $A = \{A_i(j)\}$. Due to restrictions on $\{f_{ijk}\}$ (Borisov and Volodko, submitted for publication) the rows of matrix A are linearly independent with probability 1. Obviously, given $\{\eta_k\}$, random variables $\sum_{j=1}^{\infty} A_i(j)\zeta_j$, $i = 1, 2, \ldots$, have the joint Gaussian distribution with the covariance matrix AA^T. This implies that

$$\mathbf{P}_\eta\left(\bigcap_{i=1}^{m}\left\{\left|\sum_{k=1}^{\infty} A_i(k)\zeta_k\right| < \varepsilon\right\}\right) \le \frac{\varepsilon^m}{(2\pi)^{m/2}\sqrt{\det AA^T}}. \tag{1.3}$$

Then, using the formula of total probability and (1.3), we obtain

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{i=1}^{m}\left\{\left|\sum_{k=1}^{\infty} A_i(k)\zeta_k\right| < \varepsilon\right\}\right) &\le \mathbf{P}\left(\bigcap_{i=1}^{m}\left\{\left|\sum_{k=1}^{\infty} A_i(k)\zeta_k\right| < \varepsilon\right\}\bigcap\{\sqrt{\det AA^T} > \varepsilon^{m/2}\}\right) + \mathbf{P}\left(\sqrt{\det AA^T} \le \varepsilon^{m/2}\right) \\
&= \mathbb{E}\left(\mathbf{P}_\eta\left(\bigcap_{i=1}^{m}\left\{\left|\sum_{k=1}^{\infty} A_i(k)\zeta_k\right| < \varepsilon\right\}\right)\bigcap\{\sqrt{\det AA^T} > \varepsilon^{m/2}\}\right) + \mathbf{P}\left(\sqrt{\det AA^T} \le \varepsilon^{m/2}\right) \\
&\le C(m)\varepsilon^{m/2} + \mathbf{P}\left(\sqrt{\det AA^T} \le \varepsilon^{m/2}\right).
\end{aligned}$$

This estimate leads us to the problem of the small deviations for determinants.

Regarding the topic, there are papers devoted to the small deviations for the smallest singular values of random matrices (Adamczak et al., 2012 and references therein).

Problems involving determinants of Gaussian matrices also arise in a series of works. McLennan (2002) used the expected absolute determinant of a certain Gaussian matrix to represent the number of zeros of the random multihomogeneous polynomial system. The intrinsic volume of a convex body can also be represented by $\mathbb{E}|\det M|$ where M is a random matrix with independent standard Gaussian entries (Vitale, 1991, 2008). Li and Wei (2012) obtained the estimate for $\mathbb{E}|X_1X_2\ldots X_n|$ for any centered Gaussian variables $X_1, \ldots, X_n$ with the known covariance matrix. Moreover, Li and Wei (2009) obtained the distribution of the determinant for the i.i.d. Gaussian case. In the present work, we consider matrices with jointly Gaussian (not necessarily independent) entries, but the result of Theorem 1.1 and Remark 1.2 below is not sharp for the case of i.i.d. Gaussian entries (see Remark 1.4).

Let $(\tau_{ij})_{i,j=1}^{\infty}$ be an array of centered jointly Gaussian random variables such that

$$\inf_{a_{ij}}\mathbb{E}\left(\tau_{kk} - \sum_{\min(i,j)<k} a_{ij}\tau_{ij}\right)^2 = 1. \tag{1.4}$$

Let

$$\det_n := \det\left(\tau_{ij}\right)_{i,j=1}^{n}.$$

The result is as follows.