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Characterization properties based on the Fisher information for weighted distributions

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ABSTRACT

The Fisher information on θ of the r-size weighted pdf $f_r(x;\theta)$ and its parent pdf $f(x;\theta)$ are compared leading to some characterization properties for $f(x;\theta)$. Additionally, some bounds for the Fisher information in terms of r are also presented.

1. Introduction

A biased sample arises through sampling with unequal probabilities from the original distribution. Let X be a non-negative random variable of interest such as $X \sim f(x;\theta)$, where θ is a vector of parameters. Under size-biased sampling schemes the probability of selecting a unit is proportional to x^r for some $r > 0$. Therefore the probability density function (pdf) which describes the data is of the form

$$f_r(x;\theta) = \frac{x^r f(x;\theta)}{\mu_r}$$

provided that

$$\mu_r = \int x^r f(x;\theta)dx < \infty.$$

Biased sampling is a rather common phenomenon in many fields (Zelen and Feinleib, 1969; Simon, 1980). For this reason biased sampling has been subject to many studies leading to very interesting results. Patil and Ord (1976) proved that a distribution is invariant under size-biased correction only if the distribution belongs to the log-exponential family. The reader is referred to Patil (2002) for a review on the properties of the weighted distributions and their applications to various fields.

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In this paper the Fisher information on θ contained in the r-size weighted distribution with pdf $f_r(x;\theta)$ is compared with the Fisher information on θ contained in its parent distribution with pdf $f(x;\theta)$. Recall that the Fisher information on θ is defined as

$$I(\theta) = E\left[\frac{\partial \log f(X;\theta)}{\partial\theta}\right]^2$$

and that under proper regularity assumptions (see for example Lehmann, 1983) the Fisher information can be written as

$$I(\theta) = -E\left[\frac{\partial^2 \log f(X;\theta)}{\partial\theta^2}\right]. \tag{1}$$

Throughout the paper we shall use the latter form assuming that the regularity assumptions hold. Several studies have compared the Fisher information $I(\theta)$ and $I_r(\theta)$ on θ related to $f(x;\theta)$ and $f_r(x;\theta)$ pdfs respectively. Patil and Taillie (1987) compared $I(\theta)$ and $I_r(\theta)$ within the frame of exponential families of distributions. Bayarri and DeGroot (1987a,b) investigated the Fisher information when the weight is the indicator function $w(x) = I_A$ where A is a subset of the support of an exponential family of pdfs f. Iyengar et al. (1999) studied the Fisher information for two cases which conventionally are not regarded as weighted distributions: the kth order statistic from a sample of size m from $f(x;\theta)$ and observations from the stationary distribution of residual lifetime from a renewal process, when $f(x;\theta)$ belongs to a certain exponential family. In terms of notation, $E[X]$ and $E_r[X]$ stand for the expectation of X with respect to the pdf f and f_r respectively. The notation S_X stands for the support of X. Additionally, the well known inequality

$$E[u(X)]E[v(X)] \leq E[u(X)v(X)] \tag{2}$$

which holds for any two equimonote functions $u(X)$ and $v(X)$ of the same rv X will be used in the paper (see for example Petrov, 1995). Obviously the inequality (2) is reversed if one of the two functions $u(X)$ and $v(X)$ is decreasing and the other is increasing.

2. Main results

In this section the main results of the paper are presented. Most of the results are based on the following lemma.

Lemma 1. *Let $\phi(x)$ be a real value function such that $E[e^{rX}\phi(X)] < \infty$ for all values of r in an interval O which includes 0. If*

$$E[e^{rX}\phi(X)] = E[e^{rX}]E[\phi(X)] \tag{3}$$

holds for all $r \in O$, then $P(\phi(x) = c) = 1$, where c is a constant.

Proof. Let us define the functions $\phi^+(x) = \phi(x)$ if $\phi(x) \geq 0, = 0$ if $\phi(x) < 0$, and $\phi^-(x) = -\phi(x)$ if $\phi(x) < 0, = 0$ if $\phi(x) \geq 0$. Then $\phi(x) = \phi^+(x) - \phi^-(x)$ and both ϕ^+ and ϕ^- are non-negative functions. In terms of ϕ^+ and ϕ^- relationship (3) can be expressed as

$$E[e^{rX}\phi^+(X) - \phi^-(X)] = E[e^{rX}]E[\phi^+(X) - \phi^-(X)]$$

or equivalently as

$$E[e^{rX}(\phi^+(X) + E[\phi^-(X)])] = E[e^{rX}(\phi^-(X) + E[\phi^+X])]$$

for all $r \in O$. Dividing by $E[\phi^+(X)] + E[\phi^-(X)]$ we obtain

$$E\left[e^{rX}\frac{\phi^+(X) + E[\phi^-(X)]}{E[\phi^+(X)] + E[\phi^-(X)]}\right] = E\left[e^{rX}\frac{\phi^-(X) + E[(\phi^+X)]}{E[\phi^+(X)] + E[\phi^-(X)]}\right] \tag{4}$$

for all $r \in O$. By the uniqueness of the moment generating function (4) implies that $P(\phi^+(X) + E[\phi^-(X)] = \phi^-(X) + E[(\phi^+X)]) = 1$ which means that $P(\phi(X) = c) = 1$ where c is a constant. □

Lemma 2. *Let $\phi(x)$ be a real value function such that $E[e^{rX}\phi(X)] < \infty$ for all values of r in an interval O which includes 0. If $E[X^r\phi(X)] = E[X^r]E[\phi(X)]$ holds for all $r \in O$, then $P(\phi(x) = c) = 1$ where c is a constant.*

Proof. With the help of the transformation $X = e^Y$, the relation $E[X^r\phi(X)] = E[X^r]E[\phi(X)]$ can be expressed as

$$E_Y[e^{rY}\phi(e^Y)] = E_Y[e^{rY}]E_Y[\phi(e^Y)]$$

where $E_Y(\cdot)$ is the expected value with respect to Y. So, from Lemma 1 we have that $P(\phi(e^Y) = c) = 1$, which is equivalent to $P(\phi(x) = c) = 1$, where c is a constant. □