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The strong mixing and the selfdecomposability properties

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ABSTRACT

It is proved that infinitesimal triangular arrays obtained from normalized partial sums of strongly mixing (but not necessarily stationary) random sequences can produce as limits only selfdecomposable distributions.

Selfdecomposable probability measures (in other words, the Lévy class L distributions) form (by definition) the class of possible limiting distributions of normalized partial sums from sequences of independent (but not necessarily identically distributed) random variables, under certain natural technical assumptions on the normalizing constants. The aim of this note is to show that selfdecomposable probability measures also form the class of possible limiting distributions of normalized partial sums from (not necessary stationary) strongly mixing sequences, under the same technical assumptions on the normalizing constants. The proof will utilize the standard Bernstein blocking technique.

For normalized partial sums from *strictly stationary*, strongly mixing random sequences, with a mild natural assumption on the normalizing constants, two other classes of distributions – the stable and infinitely divisible laws – have long been known to play the same roles respectively as they do in the case of i.i.d. sequences: as possible limit laws (i) along the entire sequence of normalized partial sums, and (ii) along subsequences of normalized partial sums. For further information and references on those classic results, see e.g. Volume 1, Chapter 12 of Bradley (2007). We shall not treat the particular case of strict stationarity further here.

1. Notations and basic notions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let E be a real separable Banach space, with norm $\|\cdot\|$ and Borel sigma-algebra $\mathcal{E}$. By $\mathcal{P} \equiv \mathcal{P}(E)$ we denote the set of all Borel probability measures on E, with *the convolution operation* denoted by "$*$" and *weak convergence* denoted by "$\Rightarrow$", which make $\mathcal{P}$ a topological convolution semigroup.

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Measurable functions $\xi : \Omega \to E$ are called *Banach space valued random variables* (in short: E-valued rv's) and $\mathcal{L}(\xi)(A) := P\{\omega \in \Omega : \xi(\omega) \in A\}$, for $A \in \mathcal{E}$, is *the probability distribution of* ξ. Then for stochastically independent E-valued random variables ξ_1 and ξ_2 we have that $\mathcal{L}(\xi_1) * \mathcal{L}(\xi_2) = \mathcal{L}(\xi_1 + \xi_2)$. Also for $c \in \mathbb{R} \setminus \{0\}$ and rv ξ we define $\mathcal{L}(c\,\xi)(A) = \mathcal{L}(\xi)(c^{-1}A) =: T_c(\mathcal{L}(\xi))(A)$, for $A \in \mathcal{E}$. Similarly, for $T_c : E \to E$ given by $T_c x := cx$, we define $T_c\mu$, for $\mu \in \mathcal{P}$. Hence $T_c(\mu * \nu) = T_c\mu * T_c\nu$.

For two sub-σ-fields $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$ we define *the measure of dependence* α between them as follows:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A\in\mathcal{A}, B\in\mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

For a given sequence $\mathbf{X} := (X_1, X_2, \ldots)$ of E-valued random variables, we define for each positive integer n the dependence coefficient

$$\alpha(n) \equiv \alpha(\mathbf{X}; n) := \sup_{j\in\mathbf{N}} \alpha\Big(\sigma(X_k, 1 \le k \le j),\ \sigma(X_k, k \ge j+n)\Big), \tag{1}$$

where $\sigma(\ldots)$ denotes the σ-field generated by $(\ldots)$. We will say that a sequence $\mathbf{X}$ is *strongly mixing* (Rosenblatt, 1956) if

$$\alpha(n) \to 0 \quad \text{as } n \to \infty. \tag{2}$$

Of course, if the elements of $\mathbf{X}$ are stochastically independent then $\alpha(\mathbf{X}; n) \equiv 0$. Many known stochastic processes (including many Markov chains, many Gaussian sequences, and many models from time series analysis) have long been known to be strongly mixing; see e.g. Bradley (2007).

Suppose that for stochastically independent E-valued rv's $\xi_j, j \in \mathbf{N}$, there exist sequences of real numbers a_n and vectors $b_n \in E$ and a probability measure ν such that

$$\text{(i)}\quad a_n > 0 \quad\text{and}\quad \forall(\epsilon > 0)\quad \lim_{n\to\infty} \max_{1\le k\le n} P(\{\omega \in \Omega : a_n\|\xi_k(\omega)\| > \epsilon\}) = 0 \tag{3}$$

(the so-called *infinitesimality condition*) and

$$\text{(ii)}\quad \lim_{n\to\infty} P(\{\omega \in \Omega : a_n(\xi_1 + \xi_2 + \cdots + \xi_n)(\omega) + b_n \in B\}) = \nu(B) \tag{4}$$

for every Borel set $B \subset E$ whose boundary ∂B satisfies $\nu(\partial B) = 0$; then the measure ν is called *selfdecomposable* or a *Lévy class L distribution*.

There are two basic characterizations of the class L: *the convolution decomposition* and *the random integral representation*. The first one says that

$$[\nu \in L] \quad \text{iff} \quad [\,\forall(0 < c < 1)\,\exists\,(\nu_c \in \mathcal{P}(E))\,\nu = T_c\nu * \nu_c], \tag{5}$$

and hence the term *selfdecomposability*; cf. Jurek and Mason (1993), Theorem 3.9.2.

The second one says that

$$[\nu \in L\,] \quad \text{iff} \quad \nu = \mathcal{L}\left(\int_0^\infty e^{-t} dY_\rho(t)\right), \tag{6}$$

for some Lévy process $(Y_\rho(t), t \ge 0)$ such that $\mathcal{L}(Y_\rho(1)) = \rho$ and the log-moment $\mathbb{E}[\log(1 + \|Y_\rho(1)\|)] < \infty$; cf. Jurek and Vervaat (1983) or Jurek and Mason (1993), Theorem 3.9.3. The Lévy process Y_ρ in (6) is referred to as *the background driving Lévy process* (in short: BDLP) of the selfdecomposable probability measures ν.

Finally let us note that in terms of probability measures, (4) means that

$$T_{a_n}(\rho_1 * \rho_2 * \cdots * \rho_n) * \delta_{b_n} \Rightarrow \nu \quad \text{as } n \to \infty,$$

where $\rho_i = \mathcal{L}(\xi_i)$ for $i = 1, 2, \ldots$. For probability theory on Banach spaces we refer to Araujo and Giné (1980).

2. Strong mixing and selfdecomposability

Here is the main result of this note.

Theorem 1. *Let* $\mathbf{X} := (X_1, X_2, \ldots)$ *be a sequence of Banach space E valued random variables with partial sums* $S_n := X_1 + X_2 + \cdots + X_n$, *and let* (a_n) *and* (b_n) *be sequences of real numbers and elements in E respectively, and suppose the following conditions are satisfied:*

(i) $\alpha(n) \to 0$ *as* $n \to \infty$, *i.e. the sequence* $\mathbf{X}$ *is strongly mixing;*
(ii) $a_n > 0$ *and the triangular array* $(a_n X_j,\ 1 \le j \le n, n \ge 1)$ *is infinitesimal;*
(iii) $a_n S_n + b_n \Rightarrow \mu$ *as* $n \to \infty$ *for some non-degenerate probability measure* $\mu \in \mathcal{P}$.

Then the limit distribution μ *is selfdecomposable.*

Proof of Theorem. Our aim is to show that μ satisfies the convolution decomposition (5). The argument below is divided into a few steps/observations, some of which are quite elementary.