Statistics and Probability Letters 84 (2014) 72–80

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Intuitive approximations for the renewal function

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ARTICLE INFO

Article history:
Received 30 April 2013
Received in revised form 24 September 2013
Accepted 25 September 2013
Available online 29 September 2013

MSC:
60K05
60E10
26A12

Keywords:
Renewal function
Approximations
Regular variation
The class gamma

ABSTRACT

It is hard to find explicit expressions for the renewal function $U(x) = \sum_{n=0}^{\infty} F^{*n}(x)$. Many researchers have made attempts to find suitable approximations for $U(x)$. In this paper we present simple approximations and show that they cover many of the known results.

1. Introduction

Suppose that $X, X_1, X_2, \ldots$ are i.i.d. and nonnegative r.v. with distribution function $F(x) = P(X \le x)$ and $F(0+) = 0$. If F has a density, we denote it by $f(x)$. The tail of F is denoted by $\overline{F}(x) = 1-F(x)$. If X has a finite mean, we denote it by $\mu = EX = \int_0^\infty \bar{F}(x)dx$. We assume that F is non lattice. Let $\widehat{F}(s) = E(e^{-sX})$ denote the Laplace transform of F. Let S_n denote the partial sums $S_n = X_1 + X_2 + \cdots + X_n$, $n \ge 1$, and let $S_0 = 0$. Clearly we have $P(S_n \le x) = F^{*n}(x)$, $n \ge 0$. As usual $F^*(.)$ denote the n-fold convolution of $F(.)$ with itself, that is $F^{*0}(x) = 1$, $F^{*1}(x) = F(x)$, $F^{*n}(x) = F * F^{*(n-1)}(x) = \int_0^\infty F^{*(n-1)}(x-y)dF(y)$, $n = 2, 3, \ldots$. In what follows we will use the same notation in more general sense. If $G(.)$ is a measure on the positive half line and $g(.)$ is an arbitrary measurable function we will denote $G * g(x) = \int_0^\infty g(x-y)dG(y)$.

The renewal function $U(x)$ is given by $U(x) = \sum_{n=0}^{\infty} F^{*n}(x)$. It is well known that the renewal function satisfies the following renewal equation $U(x) = 1+U*F(x)$. It is well known that $U(x) < \infty$, for all $x \ge 0$. If $\mu < \infty$, the renewal theorem states that $U(x)/x \to 1/\mu$ as $x \to \infty$ and Blackwell's theorem shows that for any fixed $y > 0$, $U(x+y) - U(x) \to y/\mu$ as $x \to \infty$. For these results we refer to Blackwell (1948) or Feller (1971). There have been many successful attempts to obtain the rate of convergence in these results. Some authors such as Rogozin (1972) or Frenk (1983, 1987) use Banach algebra techniques. Alsmeyer (1991), Carlson (1983) or Stone (1965) used Fourier analysis. Alsmeyer (1991) and Ney (1981) used coupling methods. In Dohli et al. (2002), the authors give an overview of numerical approximations in the renewal theorems. In this paper we propose a simple and intuitive approach to approximate $U(x)$. It turns out that our approximations cover all the results

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0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.09.030

that we have found in the literature. To define our approximations, recall that the Laplace transform of $U(x)$ is given by

$$\widehat{U}(s) = \int_0^\infty e^{-sx} dU(x) = (1-\widehat{F}(s))^{-1}, \quad s > 0.$$

When $\mu = EX < \infty$, we can define the equilibrium distribution function $F_e(x)$ as

$$F_e(x) = \mu^{-1} \int_0^x \overline{F}(y)dy \quad \text{for } x \geq 0.$$

If $EX^2 < \infty$ then the mean of the equilibrium distribution is finite. We will denote $\mu_e = \int_0^\infty (1 - F_e(x))dx < \infty$. Clearly $F_e(x)$ has the density $f_e(x) = \mu^{-1}\overline{F}(x),\ x \geq 0$.

Clearly we have $\widehat{F}_e(s) = (1 - \widehat{F}(s))/(\mu s),\ s > 0$ and it follows that

$$\widehat{U}(s) = \frac{1}{\mu s \widehat{F}_e(s)} = \frac{1}{\mu s} \frac{1}{1-(1-\widehat{F}_e(s))}, \quad s > 0.$$

Using a Taylor expansion, we obtain that $\widehat{U}(s) = \frac{1}{\mu s}\sum_{n=0}^\infty (1-\widehat{F}_e(s))^n = \sum_{n=0}^\infty \widehat{T}_n(s)$. Using Newton's binomial theorem, we obtain that

$$\widehat{T}_n(s) = \frac{1}{\mu s}(1-\widehat{F}_e(s))^n = \frac{1}{\mu s}\sum_{k=0}^n C_n^k (-1)^k \widehat{F}_e^k(s).$$

It follows that

$$U(x) = \sum_{n=0}^\infty T_n(x), \quad \text{for almost all } x \geq 0, \tag{1}$$

where $T_n(x) = \frac{1}{\mu}\sum_{k=0}^n C_n^k (-1)^k \int_0^x F_e^{*k}(y)dy$. We denote by $F_e^{*k}(x)$ the $k$fold convolution of $F_e(x)$ with itself. It is tempting to use formula (1) to obtain consecutive approximations for $U(x)$. Based on (1) we propose the following approximations for the renewal function: for fixed $k \geq 0$, we set

$$U_k(x) = \sum_{n=0}^k T_n(x).$$

In the paper we consider the cases $0 \leq k \leq 3$ and show that our approximation corresponds to the approximations that have been published in many papers before.

2. The functions $T_n(x)$

2.1. Some alternative expressions

Lemma 1. *Let $x \geq 0$. We have $T_0(x) = x/\mu$ and for $n \geq 1$, we have*

$$T_n(x) = -\frac{1}{\mu}\sum_{k=1}^n C_n^k (-1)^k \int_0^x (1 - F_e^{*k}(y))dy. \tag{2}$$

Moreover, if $\mu_e < \infty$, for $n \geq 2$ we have

$$T_n(x) = \frac{1}{\mu}\sum_{k=1}^n C_n^k (-1)^k \int_x^\infty (1 - F_e^{*k}(y))dy.$$

Proof. The result for $T_0(x)$ is clear. The result for $T_n(x)$ follows because we have

$$\sum_{k=0}^n C_n^k (-1)^k = (1-1)^n = 0.$$

Now assume that $\mu_e < \infty$. Clearly we have $T_n(\infty) = -\frac{\mu_e}{\mu}\sum_{k=1}^n C_n^k (-1)^k k$. Now we use Newton's binomial formula and then take the first derivative to find:

$$(x-1)^n = \sum_{k=0}^n C_n^k x^k (-1)^{n-k} \quad \text{and} \quad n(x-1)^{n-1} = \sum_{k=1}^n C_n^k k x^{k-1} (-1)^{n-k}, \quad x \in \mathbb{R}.$$

Taking $x = 1$ this is $0 = \sum_{k=1}^n C_n^k k(-1)^{n-k} = (-1)^n \sum_{k=1}^n C_n^k k(-1)^k$ and so we find that $T_n(\infty) = 0$. The result follows.

Now we consider into more details the terms $T_k(x),\ k = 1, 2, 3$.